MATHEMATICS

ACCOUNTING PROFESSION OPTION

Senior 5 Teacher's Guide

Experimental Version

Kigali, 2023

FOREWORD

Dear Teachers,
Rwanda Basic Education Board is honoured to present the teacher's guide for Mathematics in the Accounting Profession Option. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics Subject. The Rwandan educational philosophy is to ensure that students achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, the curriculum for Accounting Profession Option was reviewed to train quality Accountant Technicians who are qualified, confident and efficient for job opportunities and further studies in Higher Education in different programs under accounting career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what students learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

High Quality Technician Accounting program is an important component of Finance and Economic development of the Rwanda Vision 2050, ***"The Rwanda We Wan****t"* that aims at transforming the country's socioeconomic status. The qualified Technicians accountant will significantly play a major role in the mentioned socioeconomic transformation journey. Mathematics textbooks and teacher's guide were elaborated to provide the mathematical operations, algebraic functions and equations, and basic statistics that are necessary to train a Technician Accountant capable of successfully perform his/her duties.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum.

The Mathematics teacher's guide provides active teaching and learning techniques that engage students to develop competences. In view of this, your role as a Mathematics teacher is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students work collaboratively with more knowledgeable and experienced people.
- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group or individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, pair and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:
The part I explains the structure of this book and gives you the methodological guidance;
The part II gives a sample lesson plan;
The part III details the teaching guidance for each concept given in the student book.
Even though this Teacher's guide contains the guidance on solutions for all activities given in the student's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR Lecturers,

Teachers from TTC and General Education and experts from different Education partners for their technical support. A word of gratitude goes also to the administration of Universities, Head Teachers and TTCs principals who availed their staff for various activities.

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Director General, REB.

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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The teacher's guide of Mathematics is composed of three parts:
The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate students while working on learning activities. More other, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competence-based curriculum for pre-primary, primary, secondary education and recently the curriculum for profession options such as TTC, Associate Nurse and Accounting options . This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering students' learning achievement and creating safe and supportive learning environment. It implies also that students have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Students develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes/ values through learning activities.

In addition to the competences related to Mathematics, students also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require students to calculate, convert,interpret, analyse, compare and contrast, etc have a common factor of developing critical thinking into students
Creativity and innovation	All activities that require students to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of production/ finance/ economic have a common character of developing creativity into students
Research and problem solving	All activities that require students to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into students.
Communication	During Mathematics class, all activities that require students to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into students.
Co-operation, interpersonal relations and life skills	All activities that require students to work in pairs or in groups have character of developing cooperation and life skills among students.
Lifelong learning	All activities that are connected with research have a common character of developing into students a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling students to become life-long students who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.
Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non-verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.

The generic competences help students deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture*. Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, students should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.	Using different charts and their interpretation, Mathematics teacher should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise students on how they can fight against them. Some examples can be given when learning statistics, powers, logarithms and the related graphical interpretation.

Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Students need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.	Using Real life models or students' experience, Mathematics Teachers should lead students to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.
Financial Education: The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.	Through different examples and calculations on interest (simple and compound interests), interest rate problems, total revenue functions and total cost functions, supply and demand functions Mathematics Teachers can lead students to discuss how to make appropriate financial decisions.
Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.	Mathematics Teachers should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.
Inclusive Education: Inclusion is based on the right of all students to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.	Firstly, Mathematics Teachers need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where students can discuss how to cater for students with special educational needs.

Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.	Through a given lesson, a teacher should: Set a learning objective which is addressing positive attitudes and values, Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; Encourage students to respect ideas from others.
Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.	With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that students learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help students with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each student. Some students process information and learn more slowly than others;

- Break down instructions into smaller, manageable tasks. Students with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a student who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both students will benefit from this strategy;
- Use multi-sensory strategies. As all students learn in different ways, it is important to make every lesson as multi-sensory as possible. Students with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

  Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each student is unique with different needs and that should be handled differently:

Strategy to help students with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that students can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The student should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the student less help;
- Let the student with disability work in the same group with those without disability.

Strategy to help students with visual impairment:

- Help students to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;

- Use tactile objects to help explain a concept;
- If the student has some sight, ask him/her what he/she can see;
- Make sure the student has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that students work in pairs or groups whenever possible;

Strategy to help students with hearing disabilities or communication difficulties

- Always get the student's attention before you begin to speak;
- Encourage the student to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help students with physical disabilities or mobility difficulties:

- Adapt activities so that students who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g the height of a table may need to be changed to make it easier for a student to reach it or fit their legs or wheelchair under
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the student has one.

Adaptation of assessment strategies:

At the end of each unit, the teacher is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the teacher is expected to do assessment that fits individual students.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.

Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and teacher's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, pair and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the student
- book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics teachers need to consider various aspects

of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.

- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of students and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Teacher has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of

written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the mathematics teacher gathers information by watching students interacting, conversing, working, playing, etc. A teacher can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the teacher has to continue observing each and every activity.

- **Questioning**

a) Oral questioning: a process which requires a student to respond verbally to questions

b) Class activities/ exercise: tasks that are given during the learning/ teaching process

c) Short and informal questions usually asked during a lesson

d) Homework and assignments: tasks assigned to students by their teachers to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby students are really engaged in the learning process.

The main teaching methods used in Mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)

- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.

- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.

- **Skills lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages students in doing things and thinking about the things they are doing. Students play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, students are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of students in active learning
• The teacher engages students through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities. • He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. • He provides supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation. • Teacher supports and facilitates the learning process by valuing students' contributions in the class activities.	• A learner engaged in active learning: • Communicates and shares relevant information with fellow students through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation); • Actively participates and takes responsibility for his/her own learning; • Develops knowledge and skills in active ways; • Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings; • Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking • Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that students are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage students to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of students' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1:

- The teacher discusses convincingly with students to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to prompt / instigate collaborative learning, to discover knowledge to be learned)

Step 2:

- The teacher let students work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the students are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of students' findings/productions**

- In this part, the teacher invites representatives of groups to present their productions/findings.
- After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of students' productions.

- **Exploitation of students' findings/ productions**

- The teacher asks students to evaluate the productions: which ones are correct, incomplete or false
- Then the teacher judges the logic of the students' products, corrects those which are false, completes those which are incomplete, and confirms those

which are correct.

- **Institutionalization or harmonization (summary/conclusion/ and examples)**

- The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.

- **Application activities**

- Exercises of applying processes and products/objects related to learned unit/ sub-unit
- Exercises in real life contexts
- Teacher guides students to make the connection of what they learnt to real life situations.
- At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, students work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow students to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE OF A LESSON PLAN

School Name: …… **Teacher's name: ……**

Term	Date	Subject	Class	Unit	Lesson Number	Duration	Class size
Term ….	…….	Mathematics	S5 Acc.	35	3 of 6	40 minutes	….
Type of Special Educational needs to be catered for and the number of students in each category			….				
Unit title	Applications of derivatives in Finance and in Economics						
Key unit competence	Apply differentiation in solving Mathematical problems that involve financial quantities such as marginal cost, revenues and profits, elasticity of demand and supply						
Title of the lesson	Minimization of the cost function						
Instructional Objectives	Given a cost function by an equation, students will be able to use derivative to find the value of the independent variable for which the cost function is minimum. Hence, they will be able to precise the minimum value of the cost function.						
Plan for this Class	This lesson will take place indoor. The teacher ensures that the students' desks are arranged in a shape allowing group discussion, and the environment is free of disturbance and free of any material not related to the lesson.						
Learning Materials	Chalkboard, calculator, manila paper, markers, pens, Mathematics note books, rulers						
References	Mathematics Student text book 5 for Accounting option						

Timing for each step	Description of teaching and learning Activity: the teacher organizes the students into groups and gives clear instruction about discussion of Learning activity 3.2.1. The students discuss the activity for a while, then follows the presentation by a student, from a group, the whole class interact, the teacher helps the students to capture the key point: the necessity of finding a proper means for solving a problem about minimization of a function in general, and minimization of the cost function, in particular, without trial and error. Individual activity, whereby the students find, from daily life, quantities to minimize, will close the lesson		Generic Competences and Cross-Cutting Issues to be addressed
	Teacher activities	**Students' activities**	

Introduction 8 minutes	• Teacher distributes the flash cards, one per group, and invites the students to discuss Learning activity 3.2.1., already written on the flash card.	• Students receive the cards and start the discussion, under the organization of the group task manager	• Students acquire positive attitude by discussing without confrontation, communication skills, initiative taking, leadership
Discovery activity, presentation and summary 20 minutes	• Teacher walks around to provide assistance where necessary, and ensures that any group member participates effectively • Teacher invites the reporter of a group to present, for the whole class, the findings of the group • Teacher invites the students to interact • Teacher invites the students through well-chosen questions to highlight the main point from the discussion • Teacher requests the students to write the Learning activity in their Mathematics note book	• Students participate effectively in the discussion, the group reporter writes down, on a manila paper, the findings of the group • The group reporter presents the findings, with the support of all group members • The whole class interacts, in an organized way • Students come with the following: -What minimization is -Necessity of finding proper means to solve a problem about minimization • The students copy, from the flash cards, the Learning activity in their Mathematics note book	**Skills:** **Students** acquire knowledge, skills, positive attitude by discussing without confrontation, communication skills, initiative taking, team work spirit, value the view of others'; Problem solving; Financial education: good management of money, through minimizing the expenditure, the cost
3.Conclusion of the lesson			
Application activity and task for the next lesson 12 minutes	Teacher requests students to find, from daily life, in Economics field, examples of situations where the minimization is necessary Teacher requests students to think about means of solving a minimization problem and make research in the library, or internet	Students find examples of situations where minimization is necessary The students keep in mind the task for the next lesson	

PART III: UNIT DEVELOPMENT

Unit 1 MATRICES AND DETERMINANTS

1.1. Key unit competence

Use matrices and determinants notations and properties to solve simple production, financial economical, and mathematical related problems

1.2. Prerequisite

The students will perform well in this unit if they have a good background on:

- The use of simple and correct terminology in the description of a fact;
- Carrying out numerical calculations correctly;
- Classifying a group of quantities according to specific criteria.

1.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all students while teaching and learning
- **Peace and value Education**: During group activities, the teacher will encourage the students to help each other and to respect opinions of their classmates.
- **Gender:** Give equal opportunities to both girls and boys to participate actively in all learning and application activities from the beginning to the end of the lesson.

1.4 Guidance on introductory activity

- Form small groups of students, guide them to work on the introductory activity;
- Through class discussions, let students think of different possible criteria to classify given data and justify their validity to the entire class;
- The teacher should walk around to all groups and provide pieces of advice where necessary;
- After a given time, invite students to present their findings and

harmonize them.

- Try to arouse students' curiosity about the content of this first unit.

1.5 List of lessons

Headings	#	Lesson title/ sub-headings	Learning objectives	Number of periods
1.1. Generalities on matrices.	Introductory activity		Arouse the curiosity of students on the content of unit 1.	1
	1.	Definitions and notations	Apply definition of matrices to determine types and order of matrices	
	2.	Equality of matrices	Apply the conditions for two matrices to be equal in solving related problems	1
1.2 Operations on matrices	3.	Addition and subtraction	Perform addition and subtraction on matrices of order less than or equal to 3.	1
	4.	Scalar multiplication	Perform scalar multiplication of matrices	1
	5.	Multiplication of matrices, for $n \leq 3$	Perform multiplication on matrices of order less than or equal to 3.	2
	6.	Inversion of matrices	Determine the inverse of a matrix using row operations	3
1.3 Determinants of matrices $n \leq 3$	7.	Definition and calculation of determinants of matrices of order two or order three	Compute the determinants of order n $(n \leq 3)$ using expansion by cofactors	2
	8.	Properties of determinants	Evaluate the properties of determinants of matrices	2

1.4 Applications of matrices and determinants	9.	Inverse of a square matrix for $n \le 3$	Determine the inverse of a matrix of order less than or equal to 3.	3
	10.	Solving linear simultaneous equations using inverse of a matrix	Apply the inverse matrix and multiplication of matrices to solve systems of linear equations	3
	11.	Solving system of linear equation using Cramer's rule	Apply the determinants in solving systems of linear equations	3
1.5 End unit assessment				2

Answer to Introductory activity

From the analysis of the table, students will come with the following answers:

a) 450,000

b) 10,000

Lesson 1 : Definitions and notations

a) Learning objectives:

Apply definition of matrices to determine types and order of matrices

b) Teaching resources:

the following are possible resources that the teacher and the students can use in the teaching –learning process:

- Calculator;
- Poster;
- Manilla paper;
- Student's book;
- Eventual reference text book about matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can perform numerical calculations correctly, mentally or using a calculator.
- They can sort easily objects according to given criteria.

- They have good background on tables with double entries.

d) Learning activities

- Invite students to work in groups and do the learning activity 1.1.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work, verify and identify groups with different working steps;
- Through well-chosen questions, bring the students to discover the components of a matrix.
- Give time for individual thinking, then allow the students to exchange their view, under your vigilance.
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonise the findings from presentations of students.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices
- After this step, guide student-teachers to do the application activity 1.1.1 and evaluate whether lesson objectives are achieved

Answers of learning activity 1.1.1

a) - Type of shirts;
-Week period

b) $\begin{pmatrix} 12 & 8 & 5 \\ 9 & 3 & 0 \end{pmatrix}$

c) A matrix consists of entries arranged in rows and columns

e) Answers of application activity 1.1.1.

1) a) 2×3
b) 1×3
c) 2×1

2) $M=\begin{pmatrix} 7 & 15 \\ 4 & 9 \end{pmatrix}$, where rows stand for weeks, and columns stand for men's shoes and ladies' shoes.

Lesson 2 : Equality of matrices

a) Learning objectives:

Apply the conditions for two matrices to be equal in solving related problems

b) Teaching resources:

The following are possible resources that the teacher and the students can use in the teaching –learning process about equality of matrices

- Calculator;
- Manilla paper;
- Student's book;
- Eventual reference text book about matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can easily solve simple equations;
- From lesson1 of this unit, they understood the concept of order of a matrix, the rows and columns of a matrix;
- They can perform numerical calculations correctly, mentally or using a calculator;
- They can discover easily similarity and difference between two objects;
- They have good English background on terminologies, such as "corresponding".

d) Learning activities

- Invite students to work in groups and do the learning activity 1.1.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work, verify and identify groups with different working steps;
- Through well-chosen questions, bring the students to discover the conditions for two matrices to be equal;
- Give time for individual thinking, then allow the students to exchange their view, under your vigilance.
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a teacher, harmonise the findings from presentations of students.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices
- After this step, guide student-teachers to do the application activity 1.1.2 and evaluate whether lesson objectives are achieved

Answers of learning activity 1.1.2

a) The three matrices are of the same order 2×2

b) The corresponding entries are the same

c) The nature of elements is the same in matrices B and C

d) The matrices B and C are equal

e) For two matrices to be equal, the entries of the two matrices must be the same, and the corresponding entries must be equal.

f) Answers of application activity 1.1.2.

1.a) $A \neq B$;

b) $A = B$

2. $x = 1$ or $x = \frac{-1}{2}$; $y = \frac{1}{3}$; $z = -5$; $t = 0$ or $t = -1$

Lesson 3 : Addition and subtraction of matrices

a) Learning objectives:

Perform addition and subtraction on matrices of order less than or equal to 3.

b) Teaching resources:

The following are possible resources that the teacher and the students can use in the teaching –learning process about addition and subtraction of matrices:

- Calculator;
- Manilla paper;
- Student's book;
- Eventual reference text book about matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can easily solve simple equations;
- From lesson 1 of this unit, they understood the concept of order of a matrix, the rows and columns of a matrix;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Invite students to work in groups and do the learning activity 1.2.1 found in their Mathematics Student books;
- Through well-chosen questions, bring the students to discover the conditions for two matrices to be added;
- Give time for individual thinking, then allow the students to exchange their view, under your vigilance.
- As students are working in groups, overcome their possible challenge by providing relevant assistance.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the sum and difference of matrices.
- After this step, guide student-teachers to do the application activity 1.2.1 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.2.1

a) S and T have the same order 2×3

b) $\begin{pmatrix} 13 & 13 & 15 \\ 18 & 30 & 30 \end{pmatrix}$

c) - The same order;

-Adding the corresponding entries

e) Answers of application activity 1.2.1.

1.a) No, they cannot be added because they have different orders;

b) $A+B=\begin{pmatrix} 12 & 46 \\ 22 & 9 \end{pmatrix}; A-B=\begin{pmatrix} -10 & -12 \\ 6 & -3 \end{pmatrix}$

2.a) Addition;

b) $\begin{pmatrix} 894 & 812 \\ 971 & 799 \end{pmatrix}$

c) i) $1,611$ day scholars;

ii) 894 girls are boarders.

Lesson 4 : Scalar multiplication

a) Learning objectives:

Perform scalar multiplication of matrices

b) Teaching resources:

the following are possible resources that the teacher and the students can use in the teaching –learning process about scalar multiplication of matrices:

- Calculator;
- Manilla paper;
- Student's book;
- Eventual reference text book about matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can easily solve simple equations;
- From lesson 1 of this unit, they understood the concept equality of matrices;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Use example from daily life, such as the VAT (value added tax), to introduce the lesson;
- Invite students to work in groups and do the learning activity 1.2.2 found in their Mathematics Student books;
- Through well-chosen questions, bring the students to discover how to multiply a matrix by a scalar;
- Let students think individually, first, then allow the students to exchange their view in their respective groups, under your vigilance.
- Encourage each member to participate actively in the group;
- As students are working in groups, overcome their possible challenge by providing relevant assistance.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the sum and difference of matrices.
- After this step, guide student-teachers to do the application activity 1.2.2 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.2.2

a) By multiplying the item by 0.18

b) Multiplying the matrix by 1.18

c) Each entry of the matrix is multiplied by 1.18

d) $M' = \begin{pmatrix} 150\times1.18 & 120\times1.18 & 300\times1.18 \end{pmatrix} = \begin{pmatrix} 177 & 141.6 & 354 \end{pmatrix}$

e) Answers of application activity 1.2.2

1.a) $4A-5B=\begin{pmatrix} 22 & 11 \\ -5 & -15 \end{pmatrix}$

b) $2(A+B)=\begin{pmatrix} 2 & 10 \\ 20 & 6 \end{pmatrix}$

2. $D=\begin{pmatrix} 30 & 42 \\ 38 & 56 \end{pmatrix}$

Note: After doubling the number of each item to be sold, the number of customers is going to be increased.

Lesson 5 : Multiplication of matrices

a) Learning objectives:

Perform multiplication on matrices of order less than or equal to 3.

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing multiplication of matrices

c) Prerequisites:

Students will perform better in this lesson if:

- They can easily solve simple equations;
- They can multiply two matrices;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Invite students to work in groups and do the activity 1.2.3 found in their Mathematics Student books
- Use example from daily life, such as shopping in different supermarkets, to introduce the lesson;
- Encourage each member to participate actively in the group;
- Ensure that all the students are given opportunity to communicate through presentation of the findings to the whole class;
- As a teacher, harmonize the findings from presentation: Two matrices A and B can be multiplied together if and only if **the number of columns of A is equal to the number of rows of B.** $A_{m\times n} \times B_{m\times p} = M_{m\times p}$
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the product of matrices of order 2 and order 3.
- After this step, guide student-teachers to do the application activity 1.2.3 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.2.3

a) The number of columns of M equals the number of rows of P

b) Shopping bill of Agnes:

In supermarket $S_1: 1(1800) + 3(2500) + 2(1500) = 12300$ FRW
In supermarket $S_2: 1(1600) + 3(2000) + 2(1200) = 10000$ FRW

Shopping bill of Agnes:
In supermarket $S_1: 2(1800) + 4(2500) + 3(1500) = 18100$ FRW
In supermarket $S_2: 2(1600) + 4(2000) + 3(1200) = 14800$ FRW

$C = \begin{pmatrix} 12300 & 10000 \\ 18100 & 14800 \end{pmatrix}$, where row one shows the bills for Agnes in supermarkets S_1 and S_2;

row two shows the bills for Agnes in supermarkets S_1 and S_2

c) Two rows and two columns

d) $M.P = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix} . \begin{pmatrix} 1800 & 1600 \\ 2500 & 2000 \\ 1500 & 1200 \end{pmatrix} = \begin{pmatrix} 12300 & 10000 \\ 18100 & 14800 \end{pmatrix}$

e) Answers of Application activity 1.2.3.

1. a) A and B are not conformable for multiplication

b) $A.B = \begin{pmatrix} 29 & 22 \end{pmatrix}$

2. $M.N = \begin{pmatrix} 13 & 5 \\ 64 & 20 \end{pmatrix}$

Lesson 6 : Inversion of matrices

a) Learning objectives:

Determine the inverse of a matrix using row operations

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing inversion of matrices

c) Prerequisites:

Students will perform better in this lesson if:

- They can easily solve simple equations;
- They can multiply two matrices;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Invite students to work in groups and do the activity 1.2.4 found in their Mathematics Student books;
- From the product of two matrices yielding to the unit matrix, introduce the concept of the inverse of a matrix;

- Suggest, for discussion by students, many simple examples for the students to grasp the technique of finding the inverse of a matrix;
- Encourage each member to participate actively in the group;
- Ensure that all the students are given opportunity to communicate through presentation of the findings to the whole class;
- As a teacher, harmonize the findings from presentation by helping students to realise that calculating matrix inverse of matrix A, is to find matrix A^{-1} such that $A.A^{-1} = A^{-1}.A = I$,where I is identity matrix.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the inverse of a matrix of order two and order three.
- After this step, guide student-teachers to do the application activity 1.2.4 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.2.4

a) $A.B = \frac{1}{5}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{pmatrix}.\begin{pmatrix} 3 & 2 & 2 \\ 3 & 2 & -3 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$;

order: 3×3

e) Each entry on the main diagonal (from top left to bottom right) is 1 and any entry outside the main diagonal is 0

f) In (ii) and (iii)

g) $M.X = I$, where I is the identity matrix.

h) Answers of Application activity 1.2.4.

i)

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; the augmented matrix is $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$

b) Applying the following row operations:

(1): $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

(2): $R_1 \to R_1 - R_3$ from the result in (1)

(3): $R_1 \to R_1 - R_2$ from the result in (2), we find:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

c) $M^{-1}=\begin{pmatrix}3 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$

d) $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}.\begin{pmatrix}3 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}3 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}.\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Lesson 7 : Definition and calculations of determinants of orders 2×2 and 3×3

a) Learning objectives:

Find the value of the determinant of order n $(n\le 3)$ using expansion by cofactors

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing determinants of matrices

c) Prerequisites:

Students will perform better in this lesson if:

- They can perform numerical calculations correctly, mentally or using a calculator
- They can calculate determinants 2×2 without problems.

Knowing to perform elementary operations using calculator will ensure the students to perform better in this lesson

d) Learning activities

- Through learning activity 1.3.1, bring students to distinguish between singular matrix and non-singular matrix;
- Move around in class for facilitating students where necessary and give more clarification on eventual challenges they may face during the calculation of the determinant 2×2;
- After defining the determinant of order 3×3, ask the students if it is easy to master the sum and difference of six terms, each with3 factors.
- Let them work out to discover the necessity of mnemonic techniques for calculation of a 3×3 determinant.

– Ensure that the discussion is done without confrontation.

– Invite one member from each group with different working steps to present their work where they must explain the working steps;

– As a teacher, harmonize the findings from presentation

– As a teacher, use different probing questions, guide the students to explore the content and examples given in the student's book, and bring them to master Sarrus' method and expansion by cofactors.

– After this step, guide student-teachers to do the application activity 1.3.1 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.3.1

1. a) Each entry is multiplied by 4

   b) The entries of a row or column are scalar multiples of the corresponding entries of another

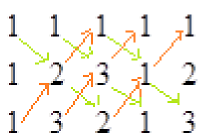
   row or column

2. Matrix A is singular

e) Answers of Application activity 1.3.1.

1. $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a matrix (a table of values in array form) but, $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is a determinant, the number $ad - bc$

2. Sarrus' method:



$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = (4+3+3)-(2+9+2) = -3$$

Expansion by cofactors:

Expanding, for example along the first row:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (4-9)-(2-3)+(3-2)-3$$

3. a) $\det X = \begin{vmatrix} 5 & 0 \\ -3 & 9 \end{vmatrix} = 45 \neq 0$; X is not singular.

   b) $\det X = \begin{vmatrix} 4k & -20k \\ -k & 5k \end{vmatrix} = 20k^2 - 20k^2 = 0$; X is singular

Lesson 8 : Properties of determinants

a) Learning objectives:

Use the properties of determinants of matrices to simplify the evaluation of a determinant

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing determinants of matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can easily calculate the determinant by using the Sarrus' method or expansion by cofactor;
- They have mastered the multiplication of matrices
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Guide the students through learning activity 1.3.2 to discover the properties of a determinant;
- Give for students to practice, many examples.

- As they are discussing the problems, check the participation of each member in his/her group.;
- Help students to elaborate the main points of the lesson
- Guide the students to do the application activity 1.3.2 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.3.2

1. a) $\det A = 0$;

b) $\det B = 0$

2. a) rows 1 and 2 of matrix A are interchanged to obtain matrix B

b) $\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} = -1$; $\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} = 1$

c) We conclude that the determinant of the matrix B is the opposite of the determinant of the matrix

e) Answers of Application activity 1.3.2.

1. a) False

b) True

2. a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

Lesson 9 : Inverse of a matrix

a) Learning objectives:

Determine the inverse of a square matrix of order less than or equal to 3.

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing determinants of matrices

c) Prerequisites:

Students will perform better in this lesson if:

- They can easily perform elementary row operations;
- They have mastered the multiplication of matrices
- They are able to calculate the determinant of a matrix of order two and order three
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Guide students, through learning activity 1.4.1 to use elementary row operations to find the inverse of a matrix.
- Then guide the students of finding the inverse of a matrix by using the method of cofactors.
- Ask the students to compare the two results
- From the two results, ask the students to conclude about how to find the inverse of a matrix
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to apply properties of the inverse of a matrix of order 2 and order 3.
- After this step, guide student-teachers to do the application activity 1.4.1 and evaluate whether lesson objectives are achieved.

Answers of learning activity 1.4.1

a) $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$

b) The inverse is $A^{-1} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

c) i) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

ii) $C = \begin{pmatrix} +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

iii) $Adj(A) = C^T = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

iv) $X = \frac{1}{\det A} Adj(A) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

d) $A^{-1} = X$

e) Answers of Application activity 1.4.1.

1) $a)\quad A^{-1} = \frac{1}{10}\begin{pmatrix} 2 & -6 \\ -5 & 20 \end{pmatrix}$

$b)\quad M^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 2 & -8 \\ 4 & 6 & -19 \\ -5 & -5 & 20 \end{pmatrix}$

2) i) $A^{-1}=\frac{1}{11}\begin{pmatrix}2 & 5 & -14\\ 1 & -3 & 4\\ -1 & 3 & 7\end{pmatrix}$, $B^{-1}=\begin{pmatrix}-\frac{5}{12} & \frac{1}{4} & -\frac{13}{12}\\ -\frac{3}{4} & \frac{1}{4} & -\frac{3}{4}\\ \frac{1}{3} & 0 & \frac{2}{3}\end{pmatrix}$

ii) $\left(A^{-1}\right)^{-1}=\begin{pmatrix}3 & 7 & 2\\ 1 & 0 & 2\\ 0 & 1 & 1\end{pmatrix}$

iii) $\left(10A\right)^{-1}=\frac{1}{110}\begin{pmatrix}2 & 5 & -14\\ 1 & -3 & 4\\ -1 & 3 & 7\end{pmatrix}$

Lesson 10 : Solving simultaneous linear equations by using inverse of matrix

a) Learning objectives:

Use the inverse matrix and the multiplication of matrices to solve systems of linear equations.

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing determinants of matrices

c)Prerequisites:

Students will learn easily in this lesson, if they have a good background on:

- definition, size and types of matrices
- operations (Addition, subtraction and multiplication) on matrices.
- how to calculate the determinant of a matrix of order 2and order 3.
- how to calculate the inverse of the matrix

d)Learning activities

- Through well-chosen questions, help the students to model the problem in learning activity 1.4.2 by matrices and simultaneous linear equations
- From the product of two matrices, help the students using inverse matrix to make a matrix the subject of the formula.
- Finally, use the product and the equality of matrices to draw the conclusion.
- As students are discussing in groups, circulate to provide assistance, where necessary.
- Let a student chosen at random present his/her findings,
- Finally, help the students to capture the main points of the lesson

Answers of learning activity 1.4.2

a) $\begin{cases} 4x+3y=53 \\ 11x+y=37 \end{cases}$

b) $A=\begin{pmatrix} 4 & 3 \\ 11 & 1 \end{pmatrix}; B=\begin{pmatrix} 53 \\ 37 \end{pmatrix}; X=\begin{pmatrix} x \\ y \end{pmatrix}$

c) $\begin{pmatrix} 4 & 3 \\ 11 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 53 \\ 37 \end{pmatrix}$

d) $A^{-1}=-\frac{1}{29}\begin{pmatrix} 1 & -3 \\ -11 & 4 \end{pmatrix};$

$-\frac{1}{29}\begin{pmatrix} 1 & -3 \\ -11 & 4 \end{pmatrix}\begin{pmatrix} 53 \\ 37 \end{pmatrix}=\begin{pmatrix} 2 \\ 15 \end{pmatrix};$

e) $\begin{cases} x=2 \\ y=15 \end{cases}$

e) Answers of Application activity 1.4.2.

a)the system is equivalent to $\begin{pmatrix} 4 & 7 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 34 \\ 11 \end{pmatrix}$

Let $A=\begin{pmatrix} 4 & 7 \\ 3 & -2 \end{pmatrix}$. Then $A^{-1}=\frac{-1}{29}\begin{pmatrix} -2 & -7 \\ -3 & 4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}=\frac{-1}{29}\begin{pmatrix} -2 & -7 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 34 \\ 11 \end{pmatrix}=\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$$\begin{cases} x = 5 \\ y = 2 \end{cases}$$

b) The system can be expressed as $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 2 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 1 & 3 & -1 \end{pmatrix}$; then $A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 1 & 3 \\ 1 & -2 & 5 \\ 8 & -5 & 7 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5 & 1 & 3 \\ 1 & -2 & 5 \\ 8 & -5 & 7 \end{pmatrix}\begin{pmatrix} -2 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{11} \\ \frac{-6}{11} \\ \frac{-37}{11} \end{pmatrix}$$

Lesson 11 : Solving simultaneous linear equations by Cramer's rule

a) Learning objectives:

Determine whether a system is a Cramer's system or not, and solve the system by Cramer's rule

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Poster,
- Manilla paper,
- Student's book;
- Any reference text book containing determinants of matrices

c) Prerequisite:

Students will perform better in this lesson if:

- They can easily solve simple simultaneous linear equations by elimination method;
- They have mastered the calculation of determinants;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Through well-chosen questions, bring the students to derive the Cramer's rule
- Provide, for discussion by students, many simple examples for the students to master the Cramer's rule;
- Ensure each member participates actively in the group;
- Give to all the students the same opportunity to communicate through presentation of the findings to the whole class;

Answers of learning activity 1.4.3

1.a) $\begin{array}{l} b'ax+b'by=b'c \\ -ba'x-bb'y=-bc' \end{array}$

b) $(ab'-ba')x=cb'-bc'$

c) $x=\dfrac{cb'-bc'}{ab'-ba'}$;

d) $x=\dfrac{\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}$

2. a) $\begin{array}{l} -a'ax-a'by=-a'c \\ aa'x+ab'y=ac' \end{array}$

b) $(-a'b+ab')\,y=-a'c+ac'$

c) $y=\dfrac{ac'-ca'}{ab'-ba'}; ab'-ba'\neq 0$

d) $y=\dfrac{\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}}{\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}}$

3. $x=\dfrac{\begin{vmatrix} 4 & 2 \\ 14 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}}=2\,;\; y=\dfrac{\begin{vmatrix} 3 & 4 \\ 5 & 14 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix}}=-1$

e) Answers of Application activity 1.4.3.

a) $x=\frac{\begin{vmatrix}34 & 7\\ 11 & -2\end{vmatrix}}{\begin{vmatrix}4 & 7\\ 3 & -2\end{vmatrix}}=5\,;\, y=\frac{\begin{vmatrix}4 & 34\\ 3 & 11\end{vmatrix}}{\begin{vmatrix}4 & 7\\ 3 & -2\end{vmatrix}}=2$

b) $x=\frac{\begin{vmatrix}-2 & -2 & 1\\ 7 & 1 & -2\\ 2 & 3 & -1\end{vmatrix}}{\begin{vmatrix}1 & -2 & 1\\ 3 & 1 & -2\\ 1 & 3 & -1\end{vmatrix}}=\frac{3}{11}\,;\, y=\frac{\begin{vmatrix}1 & -2 & 1\\ 3 & 7 & -2\\ 1 & 2 & -1\end{vmatrix}}{\begin{vmatrix}1 & -2 & 1\\ 3 & 1 & -2\\ 1 & 3 & -1\end{vmatrix}}=-\frac{6}{11}\,;\, z=\frac{\begin{vmatrix}1 & -2 & -2\\ 3 & 1 & 7\\ 1 & 3 & 2\end{vmatrix}}{\begin{vmatrix}1 & -2 & 1\\ 3 & 1 & -2\\ 1 & 3 & -1\end{vmatrix}}=\frac{-37}{11}$

1.6. Summary of unit 1

Unit 1 dealt with matrices and determinants. We studied how data can be recorded using matrices, and how information from the field of Economics and Finance can be drawn from a matrix. Different operations can be performed on matrices. We ended by determinants as means in solving problems, such as simultaneous linear equations, that can be modeled by matrices.

Therefore,

- A **matrix** is a rectangular arrangement of numbers, in rows and columns, within brackets $(\ )$ or $[\ ]$. A matrix is denoted by a capital letter: A, B, C, … .
- The numbers in the matrix are called **entries** or **elements.**
- If matrix A has n rows and p columns, then we say that the matrix A is of **order** $n\times p$, read n by p, where the product $n\times p$ is the number of entries in the matrix.
- If A is a matrix of order $n\times p$, then A can generally be written as $A=\left(a_{ij}\right)$,where i and j are positive integers, and; $1\le i\le n\,;1\le j\le p$.
- A matrix with only one row is said to be a **row matrix;** that is a matrix of order $1\times p$. For example, $\begin{pmatrix}2 & 4 & 7\end{pmatrix}$ is a row matrix.
- A matrix with only one column is said to be a **column matrix;** that is a matrix of order $n\times 1$. For example, $\begin{pmatrix}1\\ 3\\ 6\end{pmatrix}$ is a column matrix.

- A **square matrix** is a matrix in which the number of rows is equal to the number of columns; that is, matrix A of order $n \times p$ is a square matrix if and only if $n = p$;

In this case, instead of saying a matrix of order $n \times n$, we, sometimes, simply say a matrix of order n.

If $A = (a_{ij})$ is a square matrix of order 2, then i and j assume values in the set $\{1,2\}$. Therefore, $(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

In the same way, if $A = (a_{ij})$ is a square matrix of order 3, then i and j assume values in the set $\{1,2,3\}$. Therefore, $(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

- $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, *then* $\begin{pmatrix} a_{11} = b_{11} & a_{12} = b_{12} & a_{13} = b_{13} \\ a_{21} = b_{21} & a_{22} = b_{22} & a_{23} = b_{23} \\ a_{31} = b_{31} & a_{32} = b_{32} & a_{33} = b_{33} \end{pmatrix}$

- If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *and* $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, *then*

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

$$A.B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} . \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{33} + a_{33}b_{33} \end{pmatrix}$$

- If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, *then* $kA=\begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$

- If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, *then* $A^T=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

The transpose of a matrix A of order $n\times p$ is the matrix denoted A^T whose rows are the columns of A and whose columns are the rows of A.

- A matrix, in which all the entries are zeros is said to be the **null matrix** or the **zero matrix.**
- A square matrix with each element along the main diagonal (from the top left to the bottom right) being equal to and with all other elements being is said to be the **identity matrix**, it is denoted by **I;**
- For any square matrix A of order $n\times n$, and the identity matrix I of order $n\times n$, we have:
- $A.I=A$ and $I.\text{A}=A$, that is, I is the identity element for multiplication of matrices.
- Consider an arbitrary 3×3 matrix, $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the

  determinant of A is defined as follows:

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32})-(a_{31}a_{22}a_{13}+a_{32}a_{23}a_{11}+a_{33}a_{21}a_{12})$$

- Let A be a square matrix of order 2×2 or 3×3.Then the **inverse** of A Can also be calculated through the following four steps:

1) Find the determinant of A, that is $\det A$;
2) Find the matrix C of cofactors of A: each entry of A is replaced by its cofactor.
3) Find the adjoint of matrix A, denoted, $Adj(A)$: the transpose of the matrix of cofactors;

4) The inverse of matrix A is $A^{-1}=\frac{1}{\det A}Adj(A)$, where matrix A is regular, or invertible.

- To solve the two simultaneous linear equations in two unknowns x and y , **Cramer's rule** requires to go through the following steps:

1) Arrange the equations to get $\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$

Write down and calculate the **principal determinant** $D=\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}=ab'-a'b$

If $D=0$, then the system has no solution or infinitely many solutions; the system is not a Cramer's system.
If $D\neq 0$, then the system is a Cramer's system and has unique solution, proceed to the next step:

2) Write down and calculate: $D_x=\begin{vmatrix} c & b \\ c' & b' \end{vmatrix}=cb'-c'b$ and $D_y=\begin{vmatrix} a & c \\ a' & c' \end{vmatrix}=ac'-a'c$

3) Write down and calculate $x=\frac{D_x}{D}$ and $y=\frac{D_y}{D}$; the solution set of the

simultaneous equations is $S=\left\{\left(\frac{D_x}{D},\frac{D_y}{D}\right)\right\}$

In the same way, for the three simultaneous linear equations in three

unknowns, x , y and z , $\begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''x=d'' \end{cases}$

The principal determinant is $D=\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$

If $D=0$, then the system is not a Cramer's system, it may have zero solution or infinitely many solutions.
If $D\neq 0$, then the system is a Cramer's system and has unique solution; the

solution set is $S=\left\{\left(\frac{D_x}{D},\frac{D_y}{D},\frac{D_z}{D}\right)\right\}$, where

$$D_x=\begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix} \quad , \quad D_y=\begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix} \quad , \quad D_z=\begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}$$

1.7. Additional information for the teacher

It is advisable for the teacher to read ahead and master the material about the unit so as to not be embarrassed by students' questions. In particular, the teacher should explain clearly the difference between matrix and determinant, how rows and columns are counted, differentiating the main diagonal from the other diagonal, differentiating between minor and cofactor of an entry in a square matrix, the expansion by cofactor along a row or a column.

It is very necessary to help students to learn this unit and finish all given activities in the given time. The teacher may prepare many activities to students to be performed in groups at home and then present them in written form to be marked after. This strategy will help teacher to cover all required topics and concepts in this unit. The teacher will use one example and one application task while teaching and let students do the remaining tasks themselves in groups and after class.

1.8. Answers of End unit assessment

1.a) 2×3; b) 1×2

2. a) $m=-\frac{5}{2}; k=2$ b) $x=-\frac{6}{5}; y=-1; z=\frac{11}{20}; t=-3$

3.a) $\begin{pmatrix}-8 & 1\\ 5 & 8\end{pmatrix}$ b) $\begin{pmatrix}1 & -4\\ 7 & 0\end{pmatrix}$

4.a) $X=\frac{4}{5}B^{-1}.A-\frac{1}{5}I$ b) $X=\frac{7}{5}I=\frac{7}{5}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

5.a) $A.B=\begin{pmatrix}3 & 1 & 2\\ 3 & 0 & 3\\ 7 & 3 & 6\end{pmatrix}; B.A=\begin{pmatrix}7 & 1 & 2\\ 13 & 1 & 2\\ 5 & 0 & 1\end{pmatrix}$

b) $\begin{pmatrix}1 & -1 & 1\\ -3 & 2 & -1\\ -2 & 1 & 0\end{pmatrix}.\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 0 & 2 & 3\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

6.a) $A^{-1}=-\frac{1}{11}\begin{pmatrix}3 & -4\\ -2 & -1\end{pmatrix}$ b) $M^{-1}=\frac{1}{3}\begin{pmatrix}0 & 1 & 0\\ -3 & -3 & 3\\ 3 & -2 & 0\end{pmatrix}$

7.a) $x=-2$;

b)i) If $A=\begin{pmatrix}a & c\\ b & d\end{pmatrix}$, then $A^T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, and $(A^T)^T=\begin{pmatrix}a & c\\ b & d\end{pmatrix}=A$;

ii) If $A=\begin{pmatrix}a & c\\ b & d\end{pmatrix}$ and $B=\begin{pmatrix}a' & c'\\ b' & d'\end{pmatrix}$ then

$$(A.B)^T=\begin{pmatrix}aa'+cb' & ac'+cd'\\ ba'+b'd & bc'+dd'\end{pmatrix}^T=\begin{pmatrix}aa'+cb' & ba'+db'\\ ac'+cd' & cb'+dd'\end{pmatrix}=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$
$$=B^T.A^T$$

8.a) $\begin{vmatrix} 12 & 6 \\ 5 & 4 \end{vmatrix} = 18$ b) $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 63$

9.a) Row 1 or column3; $\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 24$

b) i) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 3 & 6 \end{vmatrix} = 16$ ii) $C_3 \to C_1 + 2C_2 + C_3$ iii) $\begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix} = 16$

10.a)The system can be expressed as $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$; $A^{-1} = \frac{1}{10}\begin{pmatrix} 5 & -5 & 5 \\ 2 & 6 & -4 \\ -3 & 1 & 1 \end{pmatrix}$;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 5 & -5 & 5 \\ 2 & 6 & -4 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 14 \\ 16 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

b) $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -1 \neq 0; D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 10 & 2 & 1 \end{vmatrix} = -1;$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 3 & 10 & 1 \end{vmatrix} = -2; D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 3 & 2 & 10 \end{vmatrix} = -3$$

$$x = \frac{D_x}{D} = \frac{-1}{-1} = 1; y = \frac{D_y}{D} = \frac{-2}{-1} = 2; z = \frac{D_z}{D} = \frac{-3}{-1} = 3$$

1.9. Additional activities

1.9.1. Remedial activity

1. Given the matrices $A=\begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 5 & 0\end{pmatrix}$, find $A+B$ and $A.B$

Solution:

$$A+B=\begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}+\begin{pmatrix}1 & 2\\ 5 & 0\end{pmatrix}=\begin{pmatrix}4 & 2\\ 6 & -2\end{pmatrix};$$

$$A.B=\begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}.\begin{pmatrix}1 & 2\\ 5 & 0\end{pmatrix}=\begin{pmatrix}3 & 6\\ -9 & 2\end{pmatrix}$$

2. Use matrices to solve the simultaneous linear equations:

$$\begin{cases}3x-4y=-4\\ 4x+3y=8\end{cases}$$

Solution:

The system can be expressed as $\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-4\\ 8\end{pmatrix}$

Let $A=\begin{pmatrix}3 & -4\\ 4 & 3\end{pmatrix}$, then $A^{-1}=\frac{1}{25}\begin{pmatrix}3 & 4\\ -4 & 3\end{pmatrix}$; $\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{25}\begin{pmatrix}3 & 4\\ -4 & 3\end{pmatrix}\begin{pmatrix}-4\\ 8\end{pmatrix}=\begin{pmatrix}\frac{4}{5}\\ \frac{8}{5}\end{pmatrix}$

1.9.2. Consolidation activity

1. Matrices $A=\begin{pmatrix}x & y\\ 1 & 2\end{pmatrix}$ and $B=\begin{pmatrix}0 & -1\\ 2 & 3\end{pmatrix}$ are such that $A.B=B.A$.Find the

values of x and y.

Solution:

$$A.B=\begin{pmatrix}x & y\\ 1 & 2\end{pmatrix}.\begin{pmatrix}0 & -1\\ 2 & 3\end{pmatrix}=\begin{pmatrix}2y & -x+3y\\ 4 & 5\end{pmatrix}$$

$$B.A=\begin{pmatrix}0 & -1\\ 2 & 3\end{pmatrix}.\begin{pmatrix}x & y\\ 1 & 2\end{pmatrix}=\begin{pmatrix}-1 & -2\\ 2x+3 & 2y+6\end{pmatrix}$$

$A.B=B.A$ if and only if $\begin{pmatrix}2y & -x+3y\\ 4 & 5\end{pmatrix}=\begin{pmatrix}-1 & -2\\ 2x+3 & 2y+6\end{pmatrix}$

From the equality of matrices, we find: $\begin{cases} x = \frac{1}{2} \\ y = -\frac{1}{2} \end{cases}$

2.Find the inverse of the matrix $M = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 3 & -2 \end{pmatrix}$

Solution:

$$\det M = \begin{vmatrix} 3 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 3 & -2 \end{vmatrix} = -12 \quad ;$$

Matrix of cofactors: $$C = \begin{pmatrix} +\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} \\ +\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -4 & 0 & 0 \\ 6 & -6 & -9 \\ 2 & -6 & -3 \end{pmatrix}$$

Adjoint matrix: $Adj(\mathrm{M}) = \mathrm{C}^T = \begin{pmatrix} -4 & 6 & 2 \\ 0 & -6 & -6 \\ 0 & -9 & -3 \end{pmatrix}$

$$M^{-1} = -\frac{1}{12}\begin{pmatrix} -4 & 6 & 2 \\ 0 & -6 & -6 \\ 0 & -9 & -3 \end{pmatrix}$$

1.9.3. Extended activity

The table below shows, for three related markets, the demand function and the corresponding supply function:

	Demand Q_d	Supply Q_s
Market 1	$23 - 5P_1 + P_2 + P_3$	$-8 + 6P_1$
Market 2	$15 + P_1 - 3P_2 + 2P_3$	$-11 + 3P_2$
Market 3	$19 + P_1 + 2P_2 - 4P_3$	$-5 + 3P_3$

Find the prices P_1, P_2, P_3 for equilibrium in each market.
For equilibrium to occur, we have:

$$\begin{cases} 23-5P_1+P_2+P_3=-8+6P_1 \\ 15+P_1-3P_2+2P_3=-11+3P_2 \\ 19+P_1+2P_2-4P_3=-5+3P_3 \end{cases}$$

This is equivalent to:

$$\begin{cases} -11P_1+P_2+P_3=-31 \\ P_1-6P_2+2P_3=-26 \\ P_1+2P_2-7P_3=-24 \end{cases}$$

Solving the simultaneous equations by either method, we find:
$P_1=4; P_2=7; P_3=6$

Unit 2 DIFFERENTIATION/DERIVATIVES

2.1. Key unit competence

Solve Economical, Production, and Financial related problems using Derivatives

2.2. Prerequisite

The students will perform well in this unit if they have a good background on:

- Equation of a straight line
- The gradient (or slope) of a straight line;
- Carrying out numerical calculations correctly;
- Independent variable and dependent variable;
- Numerical functions.

2.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all students while teaching and learning
- **Peace and value Education**: During group activities, the teacher will encourage the students to help each other and to respect opinions of their classmates.
- **Gender:** Give equal opportunities to both girls and boys to participate actively in all learning and application activities from the beginning to the end of the lesson.

2.4. Guidance on introductory activity

- Form small groups of students, guide them to work on the introductory activity;
- Through class discussions, let students think of the limiting value of the dependent variable as the independent variable approaches a given value;

- The teacher should check the work of different groups and provide pieces of advice where necessary;
- After a given time, invite students to present their findings and harmonize them.
- Try to arouse students' curiosity about the content of this second unit.

2.5. List of lessons

Headings	#	Lesson title/sub-headings	Learning objectives	Number of periods
2.1. Differentiation from first principles.	Introductory activity		Arouse the curiosity of students on the content of unit 2.	2
	1.	Average rate of change	Interpret the gradient of a straight line as rate of change	
	2.	Instantaneous rate of change	Use limit to find the gradient of a curve at a point	2
2.2 Rules for differentiation	3.	Differentiation of a polynomial function	Use simple rules of derivatives to differentiate a polynomial	2
	4.	Differentiation of a product function	Identify a product and use formula to differentiate it	2
	5.	Differentiation of a power function	Identify a power and use formula to differentiate it	2
	6.	Differentiation of the composite function (the chain rule)	Resolve a function into its components and use the chain rule to find the derivative	2
	7.	Differentiation of a quotient function	Use the quotient rule to differentiate a quotient	2
	8.	Differentiation of a logarithmic function	Perform differentiation of a logarithmic function	2
	9.	Differentiation of an exponential function	Perform differentiation of exponential function	2

2.3. Applications of the derivatives	10.	Equation of the tangent To the graph of a function at a point.	Determine the gradient of a curve at a point, and then the equation of the tangent	2
	11.	Hospital's rule	Remove indeterminate cases using hospital's rule	2
2.4. End of unit assessment				2

Answer to Introductory activity

a) $y = 3x - 2$:

x_1	x_2	$\Delta x = x_2 - x_1$	y_1	y_2	$\Delta y = y_2 - y_1$	$\frac{\Delta y}{\Delta x}$
1	2	1	1	4	3	3
1	1.5	0.5	1	2.5	1.5	3
1	1.1	0.1	1	1.3	0.3	3
...	...	...	...	...	...	...

For $y = x^2 + 1$:

x_1	x_2	$\Delta x = x_2 - x_1$	y_1	y_2	$\Delta y = y_2 - y_1$	$\frac{\Delta y}{\Delta x}$
1	2	1	2	5	3	3
1	1.5	0.5	2	3.25	1.25	2.5
1	1.1	0.1	2	2.21	0.21	2.1
...	...	...	...	...	...	...

b) $\frac{\Delta y}{\Delta x}$ is called gradient (slope)

c) For (1) $\frac{\Delta y}{\Delta x}$ is constant, for (2) $\frac{\Delta y}{\Delta x}$ is variable

d) $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2$

Lesson 1 : Average rate of change

a) Learning objectives:

Interpret the gradient of a straight line as the rate of change.

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing gradient of a straight line

c) Prerequisites:

Students will perform better in this lesson if:

- They can easily find the gradient of a line through two given points;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision and discuss on the activity 2.1.1.
- Through well-chosen questions, bring the students to discover how the average rate of change in a function is calculated.
- Have students discussing in groups problems involving average rate of change.
- Encourage each member to participate actively in the group.
- Ensure that all the students are given opportunity to communicate through presentation of the findings to the whole class.
- As a teacher, you have to help students to harmonize their answers.
- Use different probing questions and guide them to explore the content and examples related to the average rate of change of a function and how to denote it.
- After this step, guide students to do the application activity 2.1.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.1.1

a) $5-2=3$

b) $P(2)=9; P(5)=78$

c) $\frac{\Delta y}{\Delta x}=\frac{78-9}{5-2}=23$

d) Rate

e) Answers of Application activity 2.1.1.

1. $\frac{107500}{100}=1075$
2. $\frac{f(4)-f(2)}{4-2}=6.125$

Lesson 2 : Instantaneous rate of change

a) Learning objectives:

Use limit to find the gradient of a curve at a point.

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing derivatives from first principles

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the average rate of change from lesson 2.1.1;
- They can evaluate the limit of a function as the independent variable approaches a given value
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Ensure that the students are organized in pairs;
- Request the students to discuss in pairs the learning activity 2.1.2.;
- As they are discussing, circulate to monitor the effective participation of each member of the group;

– After a while, request a group representative to present to the whole class the group's finding;

– Have students interact, and make, under your guidance, the summary of the lesson.

– Use different probing questions and guide them to explore the content and examples related to the instantaneous rate of change of a function.

– After this step, guide students to do the application activity 2.1.2 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.1.2

a)

x_1	2.1	2.01	2.001	etc	$x_1 \to 2$
$\Delta x = x_1 - x_0$	0.1	0.01	0.001	etc	$\Delta x \to 0$
$\Delta y = y_1 - y_0$	0.41	0.0401	0.004	etc	/////////////
$\frac{\Delta y}{\Delta x}$	4.1	4.01	4.001	etc	$\frac{\Delta y}{\Delta x} \to 4$

b) $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = b$

c) $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 6x$

e) Answers of Application activity 2.1.2

1. $2Q + 7$

2. $\frac{1}{2\sqrt{x}}$

Lesson 3 : Differentiation of polynomial functions

a) Learning objectives:

Use simple rules of derivatives to differentiate a polynomial

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing rules for differentiation of polynomials

c) Prerequisites:

Students will perform better in this lesson if:

- They have knowledge and skills on identification of polynomial functions
- They have mastered the calculation of the derivative from first principles,
- from section 2.1.2;
- From the observation of a pattern, they can predict the next value;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request the students to organize themselves into small groups;
- Present the learning activity 2.2.1.for students to discuss in groups;
- Find, by reaching each group, whether the students are able to distinguish a polynomial from another function;
- Check whether each group is comfortable with differentiation from first principles;
- Have students interact, and make, under your guidance, the summary of the lesson.
- Use different probing questions and guide them to explore the content and examples related to the differentiation of polynomial functions.
- After this step, guide students to do the application activity 2.2.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.2.1

a) $g(x)$

b) A letter(variable) raised to different positive integral powers, the powers are multiplied by constants, and finally, these products are added

c) i) $\frac{dC}{dx}=0$

ii) $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$

iii) $\frac{d}{dx}(ku)=k\frac{du}{dx}$

iv) $1; 2x; 3x^2; ...; nx^{n-1}$

d) $a_1+2a_2x+3a_3x^2+...+na_nx^{n-1}$

e) Answers of Application activity 2.2.1

a) $\frac{dz}{dt}=-24t^2+15t^4$

b) $\frac{dP}{dQ}=20Q^3+3$

Lesson 4 : Differentiation of a product function

a) Learning objectives:

Use simple rules of derivatives to differentiate a product function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing rules for differentiation of product functions

c) Prerequisites:

Students will perform better in this lesson if:

- They are able to expand the product of any two given functions.
- They have mastered differentiation of polynomials from section 2.2.1.
- From the observation of a function, they can tell whether it is a product or not.
- They can perform numerical calculations correctly, mentally or using a calculator.

d) Learning activities

- Ensure that students sitting on the same desk work together;
- Present the learning activity 2.2.2.for students to discuss in groups;
- Find, by reaching each group, whether the students are able to distinguish a product from another function;
- Check whether each group is comfortable with differentiation of products;
- Have students interact, and make, under your guidance, the summary of the lesson.
- Use different probing questions and guide them to explore the content and examples related to the differentiation of polynomial functions.
- After this step, guide students to do the application activity 2.2.2 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.2.2

a) i) $uv = -12x^3 + 27x^2 + 8x - 18$

ii) $\frac{d}{dx}(uv) = -36x^2 + 54x + 8$

b) i) $\frac{du}{dx} = -6x; \frac{dv}{dx} = 4$

ii) $v\frac{du}{dx} + u\frac{dv}{dx} = -36x^2 + 54x + 8$

c) $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

e) Answers of Application activity 2.2.2.

a) $\frac{dz}{dt} = -24t^2(4+3t^5) + 15t^4(2-8t^3) = -192t^7 + 30t^4 - 96t^2$

b) $\frac{dP}{dQ} = 20Q^3(3Q-7) + 3(5Q^4) = 75Q^4 - 140Q^3$

Lesson 5 : Differentiation of a power function

a) Learning objectives:

Use simple rules of derivatives to differentiate a power function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

– Calculator,
– Manilla paper,
– Student's book;
– Any reference text book containing rules for differentiation of power functions

c) Prerequisites:

Students will perform better in this lesson if:

– They have mastered differentiation of polynomials and products from sections 2.2.1.and 2.2.2;
– From the observation of a function, they can tell whether it is a power or not;
– They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

– Ensure that students sitting on the same desk work together;
– Present the learning activity 2.2.3.for students to discuss in groups;
– Find, by reaching each group, whether the students are able to distinguish a power from another function;
– Have students interact and make, under your guidance, the summary of the lesson.
– Use different probing questions and guide them to explore the content and examples related to the differentiation of power functions.
– After this step, guide students to do the application activity 2.2.3 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.2.3

a) $(2x+1)^3 = 8x^3 + 12x^2 + 6x + 1$; $\frac{dy}{dx} = 24x^2 + 24x + 6$

b) i) $\frac{d}{dx}(2x+1) = 2$

ii)The exponent is lowered, the new exponent is the old one minus 1, and finally this result is multiplied by the derivative of the base of the power.

c) $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

e) Answers of Application activity 2.2.3.

a) $\frac{dy}{dx} = 14(7x+8)$

b) $\frac{dy}{dx} = 12(4x+5)^2$

Lesson 6 : Differentiation of the composite function (The chain rule)

a) Learning objectives:

Resolve a function into its components and use the chain rule to find its derivative

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing rules for differentiation of power functions

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered composition of two functions;
- They are able to split a function into its components;
- From the observation of a function, they can tell whether it is a power or not;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Invite student-teachers to work in pairs and do the activity 2.2.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Find, by reaching each group, whether the students are able to split a function into its components;

- Verify and identify groups with different working steps;
- As they are discussing, concentrate on slow students for further explanation and provide assistance to groups in need.
- Invite students to present the findings, and help them to harmonize the answer.
- After presentation, the teacher will help the students to apply the chain rule formula in the provided examples for better understanding.
- After this step, guide student-teachers to do the application activity 2.2.4 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lesson.

Answers of learning activity 2.2.4

a) $\frac{dy}{dx} = 3(2x+1)^2 \times (2)$

b) i) $u(x) = 2x+1; v(u) = u^3$

ii) $\frac{du}{dx} = 2; \frac{dy}{du} = 3u^2$

iii) $\frac{dy}{du} \cdot \frac{du}{dx} = (3u^2)(2) = 6(2x+1)^2$

c) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

e) Answers of Application activity 2.2.4

a) $\frac{dy}{dx} = 14(7x+8)$

b) $\frac{dy}{dx} = 12(4x-5)^2$

Lesson 7 : Differentiation of a quotient function

a) Learning objectives:

Use the quotient rule to differentiate a quotient function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,

- Manilla paper,
- Student’s book;
- Any reference text book containing rules for differentiation of power functions

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the differentiation of product functions from lesson6 of this unit;
- From the observation of a function, they can tell whether it is a quotient or not;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Invite students to work in pairs and do the activity 2.2.5 found in their Mathematics books;
- As they are discussing, concentrate on slow students for further explanation and provide assistance to groups where it is necessary.
- Invite students to present their findings, and help them to harmonize the answer.
- After presentation, the teacher will help the students to generalize the derivative of quotient function and guide them to re-work the provided examples for better understanding.
- After this step, guide students to do the application activity 2.2.5 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lesson.

Answers of learning activity 2.2.5

a) i) Let $u = 3x^2, \quad v = 2x+1$

$y = u.v^{-1}$ becomes

$y = (3x^2)(2x+1)^{-1}$

ii) $\frac{dy}{dx} = \frac{(6x)(2x+1)-(2)(3x^2)}{(2x+1)^2}$

b) $\frac{(6x)(2x+1)-(2)(3x^2)}{(2x+1)^2}$

c)They are equal

d) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

e) Answers of Application activity 2.2.5

a) $\frac{dy}{dx}=\frac{5(7x-4)-7(5x-6)}{(7x-4)^2}=\frac{22}{(7x-4)^2}$

b) $\frac{dy}{dx}=\frac{12x^3(2+5x)-5(3x^4)}{(2+5x)^2}$

Lesson 8 : Differentiation of a logarithmic function

a) Learning objectives:

Perform differentiation of a logarithmic function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing rules for differentiation of power functions

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the properties of logarithms, from unit3 studied in senior4;
- They can state the restrictions on a logarithmic function
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Let the students organize themselves in groups under your supervision;
- Present the learning activity 2.2.6.for students to discuss in groups
- As the students are discussing, find how effective the discussion is carried out;

- Check whether each group is able to find restrictions on the independent variable in a logarithmic function
- Invite students to present their findings, and help them to harmonize the answer
- After presentation, the teacher will help the students to generalize the differentiation of logarithmic functions and guide them to re-work the provided examples for better understanding.
- After this step, guide students to do the application activity 2.2.6 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lesson.

Answers of learning activity 2.2.6

Consider function $f(x)=\ln x$

a)

x_0 / Δx	1	2	3	…
0.1	$\frac{\Delta y}{\Delta x}=\frac{\ln(1+0.1)-\ln 1}{0.1}=0.953$	$\frac{\Delta y}{\Delta x}=\frac{\ln(2+0.1)-\ln 2}{0.1}=0.4879$	$\frac{\Delta y}{\Delta x}=\frac{\ln(3+0.1)-\ln 3}{0.1}=0.3278$	
0.01	$\frac{\Delta y}{\Delta x}=\frac{\ln(1+0.01)-\ln 1}{0.01}=0.995$	$\frac{\Delta y}{\Delta x}=\frac{\ln(2+0.01)-\ln 2}{0.01}=0.4987$	$\frac{\Delta y}{\Delta x}=\frac{\ln(3+0.01)-\ln 3}{0.01}=0.3327$	
0.001	$\frac{\Delta y}{\Delta x}=\frac{\ln(1+0.001)-\ln 1}{0.001}=0.9995$	$\frac{\Delta y}{\Delta x}=\frac{\ln(2+0.001)-\ln 2}{0.001}=0.4998$	$\frac{\Delta y}{\Delta x}=\frac{\ln(3+0.001)-\ln 3}{0.001}=0.33327$	
…				

b) $f'(1)=\frac{1}{1}; f'(2)=\frac{1}{2}; f'(3)=\frac{1}{3};\ldots f'(x)=\frac{1}{x}$

e) Answers of Application activity 2.2.6.

a) $\frac{dy}{dx}=\frac{3}{x}$

b) $\frac{dy}{dx}=\frac{5}{(5x+6)\ln 10}$

Lesson 9 : Differentiation of an exponential function

a) Learning objectives:

Perform differentiation of an exponential function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing differentiation of exponential functions

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the properties of exponentials, from unit3 studied in senior4;
- They can convert from logarithms to exponentials and vice versa;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Ensure that the class is ready for learning;
- Present the learning activity 2.2.7.for students to discuss in groups;
- As the students are discussing, find how effective the discussion is carried out;
- Check whether each group member is able to convert an expression from logarithm to exponential and vice versa;
- Request the students to make the summary of the lesson.
- Use different probing questions and guide them to explore the content and examples related to the differentiation of exponential functions.

– After this step, guide students to do the application activity 2.2.7 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.2.7

1.a) $x = (\frac{1}{\ln a})\ln y$

b) $1 = (\frac{1}{\ln a}).\frac{y'}{y}$

c) $y' = (\ln a)y = a^x(\ln a)$

d) i) $(a^x)' = a^x \ln a$

ii) $(e^x)' = e^x$

2. i) $(a^u)' = u'a^u \ln a$

ii) $(e^u)' = u'e^u$

e) Answers of Application activity 2.2.7

a) $\frac{dy}{dx} = 12x^2 e^{4x^3}$

b) $\frac{dy}{dx} = 5(10^{5x+6})\ln 10$

Lesson 10 : Equation of the tangent to the graph of a function at a point

a) Learning objectives:

Determine the gradient of a curve at a point, and then the equation of the tangent

b) Teaching resources:

The following materials may be used in the teaching-learning process:

– Calculator,
– Manilla paper,
– Student's book;
– Any reference text book containing the determination of the equation of the tangent to the graph of a function at a point.

c) Prerequisite:

Students will perform better in this lesson if:

- They have a good understanding of the gradient, or the slope, of a straight line;
- They are able to find the equation of the line through a point and with a given gradient;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Ensure that the class is ready for learning;
- Present the learning activity 2.3.1 for students to discuss in groups;
- As the students are discussing, find how effective the discussion is carried out;
- Check whether each group member is able to find the equation of the line through a point and with a given gradient;
- Through guided questions, bring the students to make a summary of the main points of the lesson.
- Use different probing questions and guide them to explore the content and examples related to the equation of the tangent to the graph of a function at a point.
- After this step, guide students to do the application activity 2.3.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.3.1

1. a) $m = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

b) i) $\Delta x \to 0$

ii) $f'(x_0)$

iii) tangent

c) $y - f(x_0) = f'(x_0)(x - x_0)$

e) Answers of Application activity 2.3.1.

a) $y=-\frac{15}{e^2}x+\frac{180}{e^2}$

b) $y=-\frac{\sqrt{10}}{40}x+\sqrt{10}$

Lesson 11 : Hospital's rule

a) Learning objectives:

Remove indeterminate cases using Hospital's rule

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing the determination of the equation of the tangent to the graph of a function at a point.

c) Prerequisite:

Students will perform better in this lesson if:

- They understand well what an indeterminate case is, in the calculation of a limit;
- They are able to differentiate functions as studied in the first nine lessons of this unit;
- They can find the equation of the tangent to a curve at a point;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Ensure that the students are settled for the lesson;
- Present the learning activity 2.3.2. for students to discuss in groups;
- Draw the attention of the students on the fact that $f(x_0)=0$ and $g(x_0)=0$,or $f(x_0)=\infty$ and $g(x_0)=\infty$, for the continuation to hold
- Guide students to discover that in the neighbourhood of x_0, the graph of the function and the tangent to the graph are almost the same;
- In the discussion, ensure that they write correctly the equations of the tangents to the graphs of the two functions;
- As the students are discussing, find how effective the discussion is carried out;

– Check whether each group member is able to find the equation of the line through a point and with a given gradient;
– Invite students to present their findings, and help them to harmonize the answer
– Use different probing questions and guide them to explore the content and examples related to the Hospital's rule.
– After this step, guide students to do the application activity 2.3.2 and evaluate whether lesson objectives were achieved.

Answers of learning activity 2.3.2

1.a) $y = f'(x_0)x - x_0 f'(x_0) + f(x_0)$, where $a = f'(x_0), b = -x_0 f'(x_0) + f(x_0) = -x_0 f'(x_0)$

$y = g'(x_0)x - x_0 g'(x_0) + g(x_0)$, $c = g'(x_0), d = -x_0 g'(x_0) + g(x_0) = -x_0 g'(x_0)$

b) $\lim\limits_{x \to x_0} \dfrac{f'(x_0)x - x_0 f'(x_0)}{g'(x_0)x - x_0 g'(x_0)} = \lim\limits_{x \to x_0} \dfrac{(x - x_0) f'(x_0)}{(x - x_0) g'(x_0)}$

$$= \frac{f'(x_0)}{g'(x_0)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

c) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)}$, provided $f(x_0) = 0$ and $g(x_0) = 0$, or $f(x_0) = \infty$ and $g(x_0) = \infty$

e) Answers of Application activity 2.3.2.

a) $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{(x-1)e^x} = \dfrac{1}{2e}$

b) $\lim\limits_{x \to 1} \dfrac{e^{2x-2} - 1}{\ln(5x-4)} = \dfrac{2}{5}$

2.6. Summary of unit 2

- The **average rate of change** of function , $y = f(x)$ as the independent variable x assumes values from x_1 to x_2, is the quantity defined by
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
- The **instantaneous rate of change** of function $y = f(x)$ at $x = x_0$ is the value of $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ for $x = x_0$. Equivalently, the instantaneous rate of change of function $y = f(x)$ at x_0 is the number, $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

This number is called the **derivative** of function $y = f(x)$ at x_0; it is denoted by $f'(x_o)$. More generally, the derivative of function $y = f(x)$ is denoted by $f'(x)$ or y' or $\frac{dy}{dx}$

- **The rules for differentiation:**

1) Derivative of a constant function

If f is a constant function, $f(x) = c$, for all x then $\frac{df}{dx} = \frac{d}{dx}(C) = 0$

Example: Calculate the derivative of $f(x) = 8$

Solution: $\frac{df}{dx} = \frac{d}{dx}(8) = 0$

2) Derivative of identity function

The derivative of the identity function is the constant function 1, that is

if $f(x) = x$, $\frac{df}{dx} = \frac{dx}{dx} = 1$

3) Multiplication by a scalar

The derivative of the product of a constant real number by a function equals the product of the real number by the derivative of the function; that is:

$\frac{d}{dx}(ku) = k\frac{d}{dx}(u)$, where k is a constant and u is a function of variable x

4) Sum Rule

$\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$: **the derivative of a sum** equals the sum of derivatives of the terms, where u and v are functions of variable x;

5) The Difference Rule

$$\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

6) Derivative of a power

$\frac{d}{dx}(x^n) = nx^{n-1}$; the derivative of **the nth power** of the variable equals the product of the exponent by the (n-1)th power of the variable. From the properties above, it follows that:

$$\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

This holds for any function with power: $\frac{d}{dx}(u^n) = \text{n}u^{n-1}\frac{du}{dx}$

7) Product rule

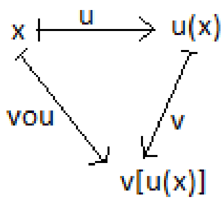
$$\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v)$$

8) Quotient rule

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, **that is,** $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

9) Differentiation of a composite function: Chain rule

Consider the following diagram:



If function $y = f(x)$ can be expressed as $y = v[u(x)]$, where $u(x)$ and $\text{v}(u)$ are functions to determine, then, it can be shown that: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$. This formula is known as the **chain rule.**

10) Differentiation of logarithmic function

The derivative of function $y = f(x) = \ln x$ is $f'(x) = (\ln x)' = \frac{1}{x}$, that is

$\frac{d}{dx}(\ln x) = \frac{1}{x}$, where $x > 0$. More generally, if $y = \ln[u(x)]$, where $u(x)$

is function of variable x, then from the chain rule, $\frac{d}{dx}[\ln u(x)] = \frac{1}{u}.\frac{du}{dx}$, that is $[\ln u(x)]' = \frac{u'}{u}$. In the same way, if $\log_a x = \frac{1}{\ln a}.\ln x$, then

$(\log_a x)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x\ln a}$, where $a > 0$ and $a \neq 1$, and $x > 0$.More generally,

If $y = \log_a u(x)$, where $u(x)$ is function of variable x, then $[\log_a u(x)]' = \frac{u'}{u\ln a}$

11) Differentiation of exponential function

The derivative of function $y = f(x) = e^x$ is $f'(x) = (e^x)' = e^x$, that is $\frac{d}{dx}(e^x) = e^x$.

More generally, if $y = e^{u(x)}$, where $u(x)$ is function of variable x, then from the chain rule, $\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)}\frac{du}{dx}$, that is $\left[e^{u(x)}\right]' = u'e^u$. In the same way, if a^x, then $\left(a^x\right)' = a^x \ln a$, where $a > 0$ and $a \neq 1$, and $x > 0$. More generally, If $y = a^{u(x)}$, where $u(x)$ is function of variable x, then $\left[a^{u(x)}\right]' = u'a^u \ln a$

12) Hospital Rule

If numerical functions $f(x)$ and $g(x)$ are such that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then
to remove the indetermination, we proceed as follows, through Hospital's rule:

1) Differentiate **separately**, the numerator and the denominator, to get $f'(x)$ and $g'(x)$;
2) Calculate $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}$
3) Then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$

Note that: - the process can be repeated if necessary;

- Hospital's rule is used only if we have indetermination $\frac{0}{0}$ or $\frac{\infty}{\infty}$

– Hospital’s rule**, is not** the quotient rule for differentiation, that is

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \neq \lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right)'$$

– Before applying Hospital’s rule, ensure that you have indetermination

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

13) Equation of Tangent line

Recall that the equation of a straight line passing through a given point $A(x_0, y_0)$ having finite slope m is given by $y - y_0 = m(x - x_0)$

or $y - f(x_0) = f'(x_0)(x - x_0$ where $f'(x_0) = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$

2.7. Additional information for the teacher.

This unit being the foundation in the sense that it has so many applications in Mathematics and in other sciences, the teacher is advised to not take this unit as granted. The examples must be well chosen and well prepared. No improvisation of data as far as modelling financial or economics problems is concerned.

Therefore, the teacher is advised to mentally prepare ahead his lessons, in order to not be embarrassed or meet a dilemma in class.

The teacher will make research in the school library or on internet so as to widen his/her knowledge.

2.8. Answers of End unit assessment

1.a) $\frac{\ln 1.2-\ln 1}{1.2-1}=0.9116$

b) $f'(x)=(\ln x)'=\frac{1}{x}; f'(1)=1$

2. a) $f'(x)=\lim_{\Delta x\to 0}\frac{3(x+\Delta x)^2-3x}{\Delta x}=6x$

b) $f'(x)=\lim_{\Delta x\to 0}\frac{\frac{-5}{5+\Delta x}-\frac{-5}{x}}{\Delta x}=\frac{5}{x^2}$

3. a) $\frac{dy}{dx}=30-x$

b) $\frac{dy}{dx}=\frac{-1}{x^2}+\frac{1}{\sqrt{x}}$

4. a) $\frac{ds}{dt}=12t^3(2t-5)+2(3t^4)=30t^4-60t^3$

b) $\frac{ds}{dt}=7t^6(t^5+11)+5t^4(t^7-4)=12t^{11}+77t^6-20t^4$

5.a) Let $u=x^2-x+2$.Then $\frac{du}{dx}=2x-1$;

The function becomes $y=\sqrt{u}$, and $\frac{dy}{du}=\frac{1}{2\sqrt{u}}$

From the chain rule, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\frac{2x-1}{2\sqrt{x^2-x+1}}$

b) Let $u=3x^4+7$.Then $\frac{du}{dx}=12x^3$

The function becomes $y=u^6$ and $\frac{dy}{du}=6u^5$

From the chain rule, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=72x^3(3x^4+7)^5$

6. a) $\frac{dQ}{dP}=\frac{(10P-9)(P^2+1)-2P(5P^2-9P+8)}{(P^2+1)^2}=\frac{9P^2-6P-9}{(P^2+1)^2}$

b) $\frac{dQ}{dP}=\frac{6(8P-5)-8(6P-7)}{(8P-5)^2}=\frac{26}{(8P-5)^2}$

7. a) $\frac{dy}{dx}=\frac{3(3x+4)^2(10x-7)}{(5x-1)^2}$

b) $\frac{dy}{dx}=\frac{-26(2x+1)}{(3x-5)^3}$

8. a) $\frac{dy}{dx}=-3(2^{1-3x}\ln 2)$

b) $\frac{dy}{dx}=\frac{3\ln(5\sqrt{x})}{x}$

9. a) $y=-6x-2$

b) $y=\frac{1}{4}x+\frac{1}{4}e^2$

10. a) $2-\frac{1}{e}$ b) $\frac{1}{2}$

2.9. Additional activities

2.9.1. Remedial activity

1. Find the derivative, with respect to x ,of:

a) $y=\frac{4x}{1-x}$

b) $y=x^2e^{-x}$

Solution:

a) $\frac{dy}{dx}=\frac{4}{(1-x)^2}$

b) $\frac{dy}{dx}=(-x^2+2x)e^{-x}$

2. Find the equation of the tangent to the graph of:

a) $y=4-2x-2x^2$ at $x=-1$

b) $y=\frac{1}{x}$ at $x=2$

Solution:

a) $y=2x+6$

b) $y=-\frac{1}{4}x+1$

2.9.2. Consolidation activity

1.Find the derivative, with respect to x ,of:

a) $y=\ln x-\frac{1}{2}\ln(1+x^2)$ when $x=2$

b) $y=3^{\sqrt{2x-1}}$ when $x=1$

Solution:

a) $\frac{dy}{dx}=\frac{1}{x}-\frac{x}{1+x^2}$

The value of the derivative at $x = 2$ is $\frac{1}{10}$

b) $\frac{dy}{dx} = \frac{3^{\sqrt{2x-1}} \ln 3}{\sqrt{2x-1}}$

The value of the derivative at $x = 1$ is $3\ln 3$

2. Calculate the following limits:

a) $\lim_{x \to 0} \frac{e^x - 1}{(x+1)e^x - 1}$

b) $\lim_{x \to 1} \frac{x - 1 - x\ln x}{(x-1)\ln x}$

Solution:

a) $\lim_{x \to 0} \frac{e^x - 1}{(x+1)e^x - 1} = \frac{0}{0}$:indeterminate case. By Hospital's rule, we have:

$$\lim_{x \to 0} \frac{e^x - 1}{(x+1)e^x - 1} = \lim_{x \to 0} \frac{e^x}{(x+2)e^x} = \lim_{x \to 0} \frac{1}{x+2} = \frac{1}{2}$$

b) $\lim_{x \to 1} \frac{x - 1 - x\ln x}{(x-1)\ln x} = \frac{0}{0}$: indeterminate case. By Hospital's rule, we have:

$$\lim_{x \to 1} \frac{x - 1 - x\ln x}{(x-1)\ln x} = -\frac{1}{2}$$

2.9.3. Extended activity

Differentiate, from first principles, the function $f(x) = \frac{x}{x^2 + 1}$

Solution:

$$\lim_{h \to 0} \frac{\frac{x+h}{(x+h)^2 + 1} - \frac{x}{x^2 + 1}}{h} = \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Unit 3 APPLICATIONS OF DERIVATIVES IN FINANCE AND IN ECONOMICS

3.1. Key unit competence

Apply differentiation in solving Mathematical problems that involve financial context such as marginal cost, revenues and profits, elasticity of demand and supply

3.2. Prerequisite

The students will perform well in this unit if they have a good background on:

- The derivatives as studied in unit 2 preceding this unit;
- The understanding of English language in order to model a problem by an equation;
- The economic and financial concepts such as cost, revenue, profit, etc.
- Carrying out numerical calculations correctly;
- Manipulation of calculators for computing data.

3.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all students while teaching and learning
- **Peace and value Education**: During group activities, the teacher will encourage the students to help each other and to respect opinions of their classmates.
- **Gender:** Give equal opportunities to both girls and boys to participate actively in all learning and application activities from the beginning to the end of the lesson.

3.4 Guidance on introductory activity

- Form small groups of students, guide them to work on the introductory activity;
- Through class discussions, let students think of how to model and solve a problem;
- The teacher should walk around to all groups and provide pieces of advice where necessary;
- After a given time, invite students to present their findings and harmonize them.
- Try to arouse students' curiosity about the content of this third unit.

3.5. List of lessons

Headings	**#**	**Lesson title/sub-headings**	**Learning objectives**	**Number of periods**
3.1. Marginal quantities	Introductory activity		Arouse the curiosity of students on the content of unit 3.	3
	1.	Marginal cost	Interpret the marginal cost as rate of change in the cost	
	2.	Marginal revenue	Interpret the marginal revenue as the rate of change in the revenue	3
3.2. Minimization and maximization of functions	3.	Minimization of the total cost function	Determine the price that will minimize the total cost function	3
	4.	Maximization of the total revenue function	Determine the price that will maximize the total revenue function	3
3.3. Price elasticity	5.	Elasticity of demand	Measure the responsiveness of demand to change in price	2
	6.	Elasticity of supply	Measure the responsiveness of supply to change in price	2
3.4. End of unit assessment				**2**

Answer to Introductory activity

a) $h = \dfrac{300}{\pi x^2}$;

b) $C(x) = 50\pi x^2 + \dfrac{12000}{x}$;

c) $x = \sqrt[3]{\dfrac{120}{\pi}} \approx 3.367$ cm; $h \approx 8,427$ cm

Lesson 1 : Marginal cost

a) Learning objectives:

Interpret the marginal cost as rate of change in the cost

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about marginal cost

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the cost function as studied in Senior4, unit2;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;
- Through well-chosen questions and discussion of learning activity 3.1.1, bring the students to discover that the instantaneous rate of change in the cost is called the marginal cost and how to calculate it;
- Encourage each member to participate actively in the group;
- Ensure that all the students are given opportunity to communicate through presentation of the findings to the whole class;
- Use different probing questions and guide them to explore the content and examples related to the Marginal cost.
- After this step, guide students to do the application activity 3.1.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 3.1.1

a) $\frac{dy}{dx} = 6x + 7$ .For $x = 3$, $\frac{dy}{dx} = 25$

b) Marginal total cost

e) Answers of Application activity 3.1.1.

1. $\frac{dC}{dx} = 4 - 2x + 6x^2$

2. $\frac{dQ}{dP} = 40 + 6P - P^2$;

For $P = 10$,the marginal product is 0

Lesson 2 : Marginal revenue

a) Learning objectives:

Interpret the marginal total revenue as rate of change in the total revenue

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about marginal revenue

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the revenue function as studied in Senior4, unit2;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;

- Through well-chosen questions and discussion of learning activity 3.1.2, bring the students to discover that the instantaneous rate of change in the total revenue is called the marginal total revenue and how to calculate it;
- Encourage each member to participate actively in the group;
- Ensure that all the students are given opportunity to communicate through presentation of the findings to the whole class;
- Use different probing questions and guide them to explore the content and examples related to the Marginal revenue.
- After this step, guide students to do the application activity 3.1.2 and evaluate whether lesson objectives were achieved.

Answers of learning activity 3.1.2

a) $100Q - Q^2$

b) $100 - 2Q$; for $Q = 11$, the value is 78

e) Application activity3.1.2.

1. The total revenue function if $R(x) = 30x - 2x^2$; the marginal revenue is $\frac{dR}{dx} = 30 - 4x$
2. The total revenue function is $R(Q) = Q^3 + 2Q^2 + Q$, the total marginal revenue is $\frac{dR}{dQ} = 3Q^2 + 4Q + 1$;For $Q = 10$, $\frac{dR}{dQ} = 341$

Lesson 3 : Minimization of the total cost function

a) Learning objectives:

Determine the price that will minimize the total cost function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about marginal revenue

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the cost function as studied in Senior4, unit2;
- They are able to differentiate simple functions as covered in unit2 of this year;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;
- Through well-chosen questions and discussion of learning activity 3.2.1, bring the students to discover how to model a problem and how to express the quantity to minimize as function of one variable;
- Ensure that the students can differentiate and set the conditions for a minimum to occur;
- Encourage each member to participate actively in the group;
- Request a student, chosen at randomly, to present the findings of his/her group to the whole class;
- Use different probing questions and guide the students to explore the content and examples related to the Minimization of the total cost function.
- After this step, guide students to do the application activity 3.2.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 3.2.1

How do I choose the base radius and the height of the can so as to pay the least money for purchasing the material?

e) Application activity 3.2.1.

f)

a) $\frac{dP}{dQ}=\frac{2Q-8}{Q^2-8Q+20}=0$ if and only if $Q=4$ and the derivative changes its sign from negative to positive; therefore, the minimum occurs for $Q=4$

b) $\frac{dP}{dQ}=Q^2-17Q+60=0$ if and only if $Q=5$ or $Q=12$

We have $\frac{d}{dQ}(Q^2-17Q+60)=2Q-17$; this value is positive for $Q=12$;

therefore, the minimum value occurs for $Q=12$

Lesson 4 : Maximization of the total revenue function

a) Learning objectives:

Determine the price that will maximize the total revenue function

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about applied extrema problems.

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the revenue function as studied in Senior4, unit2;
- They can easily solve simple equations in one unknown;
- They are able to find derivatives of simple functions as covered in unit2 of this year;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;
- Through well-chosen questions and discussion of learning activity 3.2.2, bring the students to discover how to model a problem and how to express the quantity to maximize as function of one variable;
- Ensure that the students can find the derivative of a function and set the conditions for a maximum to occur;
- Encourage each member to participate actively in the group;
- As they are discussing, concentrate on slow students for further explanation and provide assistance to groups in need

- Request a student, chosen at randomly, to present the findings of his/her group to the whole class, and help them to harmonize the answer
- Use different probing questions and guide the students to explore the content and examples related to the Maximization of the total revenue function.
- Ask students to work out examples under your guidance, and work individually application activity 3.2.2 to check the skills they have acquired.

Answers of learning activity 3.2.2

How do I choose the length and the width of the rectangular plot so as to obtain a plot with the greatest area?

e) Application activity 3.2.2.

The total revenue function is $R(Q) = 24Q - 3Q^2$

$\frac{dR}{dQ} = 24 - 6Q = 0$ if and only if $Q = 4$ .Since $\frac{d}{dQ}(24 - 6Q) = -6 < 0$, the

maximum value of the total revenue occurs for $Q = 4$, and it is equal to 48

Lesson 4 : Price elasticity of demand

a) Learning objectives:

Measure the responsiveness of demand to change in price

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about applied extrema problems.

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the demand function as studied in Senior4, unit2;
- They can easily solve simple equations in one unknown;
- They are able to find derivatives of simple functions as covered in unit 2 of this year;

- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;
- Through well-chosen questions and discussion of learning activity 3.3.1, bring the students to discover how to model a problem and how to express the quantity to maximize as function of one variable;
- Ensure that the students can find the derivative of a function and set the conditions for a maximum to occur;
- Encourage each member to participate actively in the group;
- As they are discussing, concentrate on slow students for further explanation and provide assistance to groups in need
- Request a student, chosen at randomly, to present the findings of his/her group to the whole class, and help the students to harmonize the answer
- Use different probing questions and guide the students to explore the content and examples related to the Elasticity of demand.
- Ask students to work out examples under your guidance, and work individually application activity 3.3.1 to check the skills they have acquired.

Answers of learning activity 3.3.1

If the price increases, then the quantity demanded will decrease.

e) Application activity3.3.1.

a) $\frac{dQ_d}{dP}=\frac{-2}{P}$

At $P=4$, $Q_d=\ln\frac{100}{16}=1.832;\frac{dQ_d}{dP}=\frac{-1}{2}$

The price elasticity of demand is $\varepsilon_d=\frac{dQ_d}{dP}.\frac{P}{Q_d}=\frac{-1}{2}.\frac{4}{1.832}=-1.09$

b) $\frac{dQ_d}{dP}=\frac{-20}{(P+1)^2}$

At $P=3$, $Q_d=5;\frac{dQ_d}{dP}=\frac{-20}{16}=\frac{-5}{4}$

The price elasticity of demand is $\varepsilon_d = \frac{dQ_d}{dP} \cdot \frac{P}{Q_d} = \frac{-5}{4} \cdot \frac{3}{5} = \frac{-3}{4} = -0.75$

Lesson 6 : Price elasticity of supply

a) Learning objectives:

Measure the responsiveness of supply to change in price

b) Teaching resources:

The following materials may be used in the teaching-learning process:

- Calculator,
- Manilla paper,
- Student's book;
- Any reference text book containing information about price elasticity.

c) Prerequisites:

Students will perform better in this lesson if:

- They have mastered the supply function as studied in Senior4, unit2;
- They can easily solve simple equations in one unknown;
- They are able to find derivatives of simple functions as covered in unit2 of this year;
- They can perform numerical calculations correctly, mentally or using a calculator;

d) Learning activities

- Request students to organize themselves in groups under your supervision;
- Through well-chosen questions and discussion of learning activity 3.3.2, bring the students to discover how the price elasticity of supply is calculated;
- Ensure that the students can find the derivative of a function and they can use the formula of price elasticity of supply.
- Encourage each member to participate actively in the group;
- Request a student, chosen at randomly, to present the findings of his/her group to the whole class, and help the students to harmonize the answer

- Use different probing questions and guide the students to explore the content and examples related to the Elasticity of supply.
- Ask students to work out examples under your guidance, and work individually application activity 3.3.2 to check the skills they have acquired.

e) Answers to activities

Learning activity 3.3.2.

If the price increases, then the quantity supplied will increase.

Application activity 3.3.2.

a) $\frac{dQ_s}{dP} = \frac{1}{4}$

At $P = 11, Q_s = 2; \frac{dQ_s}{dP} = \frac{1}{4}$

The price elasticity of supply is $\varepsilon_s = \frac{dQ_s}{dP} \cdot \frac{P}{Q_s} = \frac{1}{4} \cdot \frac{11}{2} = \frac{11}{8}$

b) $\frac{dQ_s}{dP} = 5e^{-0.2P} - Pe^{-0.2P}$

At $P = 4, Q_s = 20e^{-0.8}; \frac{dQ_s}{dP} = e^{-0.8}$

The price elasticity of demand is $\varepsilon_s = \frac{dQ_s}{dP} \cdot \frac{P}{Q_s} = e^{-0.8} \cdot \frac{4}{20e^{-0.8}} = \frac{1}{5}$

3.6. Summary of unit 3

In this unit, we focused on some examples of how derivatives can be used in Finance and in Economics. The following points were considered:

1) **Marginal cost**

Suppose a manufacturer produces and sells a product. Denote C(q) to be the total cost for producing and marketing q units of the product. Thus, C is a function of q and it is called the *(total) cost function*. The rate of change of C with respect to q is called the *marginal cost*, that is,

Marginal Cost $= \frac{dC}{dq}$

2) **Marginal revenue**

If $y = C(x)$ is cost of producing x units of a product, then $R(x)$,the total revenue generated by selling x units of the product, is given by $R(x) = x.C(x)$

: the product of the number of units produced by the cost of producing the units. Then, the **marginal revenue** is the instantaneous rate of change in the total revenue, that is $\frac{dR}{dx}$.

3) Minimization of the total cost function

If function $y = f(x)$ is such that $\frac{dy}{dx} = 0$ at x_0 and $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$ at x_0, then the function $y = f(x)$ has a local **minimum** at x_0; the minimum value of the function is $f(x_0)$.

Function $y = f(x)$ is said to be **increasing** for the values of x such that $\frac{dy}{dx} > 0$, and **decreasing** for the values of x such that $\frac{dy}{dx} < 0$

4) Maximization of the total revenue function

If the total cost function is $y = f(x)$, then the total revenue function is $R = xy$.

Suppose $\frac{dR}{dx} = 0$ at x_0 and $\frac{d}{dx}\left(\frac{dR}{dx}\right) < 0$ at x_0, then the total revenue function $R = xy$ has a local **maximum** at x_0; the maximum value of the total revenue function is $\mathrm{R}(x_0)$.

5) Elasticity of demand

In Economics, **price elasticity** ε_d measures the percentage change in quantity associated with a percentage change in price. If the quantity Q_d is related to price P by, $Q_d = f(P)$, then the **elasticity of demand** is defined by

$$\varepsilon_d = \frac{dQ_d}{dP}.\frac{P}{Q_d}.$$

Price elasticity of demand indicates how consumers respond to the change in the amount proposed by the producers.

If $\varepsilon_d < 0$, then Q_d and P are such that the increase in P implies the decrease in Q_d

6) Elasticity of supply

If the quantity Q_s is related to price P by $Q_s = f(P)$, then the **elasticity of demand** is defined by $\varepsilon_s = \frac{dQ_s}{dP}.\frac{P}{Q_s}$. In some cases, P is given in terms of Q_s.

In this case, start by making Q_s the subject of the formula.

The price **elasticity of supply** is defined by, $\varepsilon_s = \frac{dQ_s}{dP}.\frac{P}{Q_s}$,

Where Q_s **:** quantity supplied, and P **:** amount received from consumers.
Price elasticity of supply indicates how producers respond to the change in the amount they receive from the consumers

3.7. Additional information for the teacher.

The teacher will make research in the school library or on internet so as to widen his/her knowledge.
Examples and exercises must be realistic; therefore, the teacher is advised to choose and prepare carefully the examples to discuss.

3.8. Answers of End unit assessment

1.The total revenue function is $R = 12.5Qe^{-0.005Q}$;

$\frac{dR}{dQ} = (12.5 - 0.0625Q)\,\mathrm{e}^{-0.005Q} = 0$ if and only if $Q = 200; P = 1$

2. $\frac{d}{dx}\left(\frac{4x}{3\ln x}\right) = \frac{12(\ln x - 1)}{9\ln^2 x} = 0$ for $x = e \approx 2.718$

3.The marginal cost is $\frac{dC}{dQ} = 3Q^2 - 6Q + 15$

4.Total revenue is $R = 30Q - Q^2$

The total marginal revenue is $\frac{dR}{dQ} = 30 - 2Q$

5. $\frac{dQ}{dP} = -15$; the price elasticity of demand is $-\frac{2}{3}$

3.9. Additional activities

3.9.1. Remedial activity

1) Given the total revenue $R = 80P - 2P^2$, where P is the price of an item, obtain the marginal total revenue.

2) The total cost of commodity is given by the total cost function $C = \frac{25}{Q} + 0.1Q^2$. Find the value of Q that will minimize the total cost. Prove that for this value, the minimum cost occurs, and find the minimum total cost.

Solution:

1) The marginal total revenue is $\frac{dR}{dP} = 80 - 4P$

2) $\frac{dC}{dQ} = \frac{-25}{Q^2} + 0.2Q = 0$ if and only if $Q = 5$

$\frac{d}{dQ}\left(\frac{-25}{Q^2} + 0.2Q\right) = \frac{50}{Q^3} + 0.2$. For $Q = 5$, $\frac{d}{dQ}\left(\frac{-25}{Q^2} + 0.2Q\right) = \frac{50}{5^3} + 0.2 > 0$

.Therefore, the minimum total cost occurs for $Q = 5$, and the minimum total cost is $C = \frac{25}{5} + 0.1(5)^2 = 7.5$

3.9.2. Consolidation activity

1. The quantity demanded and the price Q_d and the price P of a commodity are related by the equation $P = 60 - 3Q_d$. Find the price elasticity of the demand when $P = 12$

2. The total demand function of a firm is given by $Q_d = 200 - 2Q$. Find the marginal total revenue

Solution:

1. We have: $Q_d = \frac{-1}{3}P + 20$;

$\frac{dQ_d}{dP} = \frac{-1}{3}$;

For $P = 12$, $\frac{dQ_d}{dP} = \frac{-1}{3}; Q_d = \frac{-1}{3}(12) + 20 = 16$;

The price elasticity of the demand is $\varepsilon_d = \frac{dQ_d}{dP}.\frac{P}{Q_d} = \frac{-1}{3}.\frac{12}{16} = \frac{-1}{4}$

2.The revenue function of the firm is $R = 200Q - 2Q^2$;

He marginal total revenue is $\frac{dR}{dQ} = 200 - 4Q$

3.9.3. Extended activity

The total cost function and the total revenue of a firm are given respectively by $C = 4 + 97Q - 8.5Q^2 + \frac{1}{3}Q^3$ and $R = 58Q - \frac{1}{2}Q^2$

a) Find the total profit
b) Find the value of Q for which the total profit is maximum, and show that for this value, the profit is, indeed, maximum
c) Find the maximum profit.

Solution:

a) The maximum profit is

$$\pi = (58Q - \frac{1}{2}Q^2) - (4 + 97Q - 8.5Q^2 + \frac{1}{3}Q^3) = -4 - 39Q + 8Q^2 - \frac{1}{3}Q^3$$

b) $\frac{d\pi}{dQ} = -39 + 16Q - Q^2 = 0$ if and only if $Q = 3$ or $Q = 13$

For $Q = 3$, we have: $\frac{d}{dQ}(\frac{d\pi}{dQ}) = \frac{d}{dQ}(-39 + 16Q - Q^2) = 16 - 2Q > 0$,

thus, the maximum does not occur for $Q = 3$

For $Q = 13$, we have: $\frac{d}{dQ}(\frac{d\pi}{dQ}) = \frac{d}{dQ}(-39 + 16Q - Q^2) = 16 - 2Q < 0$, showing that the maximum value of the total profit occurs for $Q = 13$;

c)The maximum total profit is $-4 - 39(13) + 8(13)^2 - \frac{1}{3}(39)^3 = 108.7$

Unit 4 UNIVARIATE STATISTICS AND APPLICATIONS

4.1 Key unit competence

Apply univariate statistical concepts to collect, organise, analyse, present, and interpret data to draw appropriate decisions.

4.2 Prerequisite

Students will perform well in this unit if they are familiar with statistics learned in lower secondary school (S1 and S3) and skilled in using scientific calculators.

4.3 Cross-cutting issues to be addressed

- Inclusive education (promoting education for all in teaching)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Financial education (develop sprits of saving and investments decisions)
- Gender (equal opportunity for boys and girls to participate in class)
- Standardized culture (quality of production)

4.4 Guidance on introductory activity

- Invite students to work in a group, discuss and find out the answers for the introductory activity from the student's book.
- Facilitate students' discussions and ask them to avoid noise or other unnecessary conversations.
- During discussions, let students think of different ways to solve the given problem.
- Walk around in all groups to help if necessary.
- Invite group members to present their findings and encourage boys and girls to actively participate in presentations.

- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this introductory activity, use different probing questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer for an introductory activity

Use the introductory activity to give an overview of the whole unit, the key concepts that will be discussed, and concepts such as inferential and descriptive statistics, univariate data, population, sample, mean, mode, median, quartile, variance, standard deviation, sampling and sampling methods, quantitative and qualitative data, continuous and discrete data, etc.

1. (a) The more data you have at your disposal, the better your position will be to make good decisions and take advantage of new opportunities. Good data will also give you the justification and evidence to back up these decisions so that you can feel confident explaining your reasoning. Without reliable data, you're much more likely to make mistakes and reach incorrect conclusions.

   (b) In finance, quantitative data can be collected to determine a company's business income, earnings, and revenue-generating capacity. Qualitative data can also be collected to understand customers' satisfaction with the company's products and services.

   (c) A family can record daily the money spent on each item bought; this information can help the family make a well-informed decision on the family budget. Similarly, a country needs information on the amount of money to allocate to each sector (e.g., education, security, agriculture, etc.), which can help make a national budget.

2. i) Data related to people's favorite food.

   ii) Questionnaire, interview

   iii) Mode

3. During the accounting exam, out of 10, ten students scored the following marks: 3, 5,6,3,8,7,8,4,8, 6.

a) $\frac{50}{10}=5$

b) 3 and 8

c) Among ten students, only three scored below the average mark. This indicates that students performed well in accounting exams.

4.5 List of lessons

Headings		Lesson title/sub-headings	Learning objectives	Number of periods
4.1 Basic concepts in univariate statistics.		Introductory activity	Arouse the curiosity of students on the content of unit 4.	1
	1	Statistical concepts: statistics, descriptive and inferential statistics, population, sample, statistic, and parameter.	Explain and differentiate various concepts used in statistics	2
	2	Variables and types of variables	Differentiate types of variables	2
	3	Data and types of data	Differentiate types of data	2
	4	Levels of measurement scale	Differentiate levels of measurement scale	2
	5	Sampling and sampling methods	Differentiate different sampling methods and decide which one is feasible depending on the context under study	3
4.2 Organizing and graphing data	1	Frequency table	Represent data accurately using a frequency distribution table	2
	2	Bar graph	Represent data accurately using a bar graph	2
	3	Histogram	Represent data accurately using a histogram	2
	4	Time series graph	Represent time series data using a time series graph	2
	5	Pie chart	Represent data accurately using a Pie chart	2
	6	Graph interpretation	Interpret a statistical graph.	1

4.3 Numerical descriptive measures	1	Describing data using mean, median and mode	Describe data using central measures, mean, median and mode and interpret the results	1
	2	Summarizing data using variance, standard deviation, and coefficient of variation	Summarize data using spread measures, variance, standard deviation, and coefficient of variation and interpret the results	2
	3	Determining the position of data value using quartiles	Determine the position of data value using quartiles	2
4.4 Measure of symmetry	1	Skewness	Determine the skewness of the data	1
	2	Chebyshev's theorem and Empirical rule	Apply Chebyshev's theorem and Empirical rule to determine the skewness of the data	1
4.5 Applications of univariate statistics in mathematical problems that involve finance, accounting, and economics.	1	Examples of the applications of univariate statistics in solving problems related to finance, accounting, and economics.	Apply univariate statistics in solving mathematical problems that involve finance, accounting, and economics	1
4.6 End unit assessment				1

Lesson 1 : Statistical concepts: descriptive and inferential statistics, population, sample, statistic, and parameter

a) Learning objective

Explain and differentiate various concepts used in statistics

b) Teaching resources

Student's book and other reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.1.1 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on concepts and terminologies used in statistics.
- Ask students to do the application activity 4.1.1. and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.1.1

1.
   i) The gathering, presentation, categorization, analysis, and interpretation of data.

   ii) Basically, there are two branches of statistics. They are descriptive and inferential.

   iii) Data, variable, population, mean, sample, etc

2. A company may select a sample of students before supplying and ask what students prefer between milk and fruits. Information gathered from that sample can help the company to make a well-informed decision about what more students are likely to choose between milk and fruits.

e) Answers of application activity 4.1.1

1. Suppose the scores of 100 students belonging to a specific country are available. The performance of these students needs to be examined. This data by itself will not yield any valuable results. However, by

using descriptive statistics, the spread of the marks can be obtained, thus, giving a clear idea regarding each student's performance. Now, suppose the scores of the students of an entire country need to be examined. Using a sample of 100 students, inferential statistics is used to make generalizations about the population.

2. In this example:

- The population is all senior five students attending Kiziguro secondary school.
- The sample could be all students enrolled in one section (combination, let say, Mathematics, Economics, and Geography) at Kiziguro secondary school (although this sample may not represent the entire population).
- The parameter is the average (mean) amount of money spent (excluding books) by senior five students at Kiziguro secondary school: the population mean.
- The statistic is the average (mean) amount of money spent (excluding books) by senior five students in the sample.

Lesson 2 : Variables and types of variables

a) Learning objective: Differentiate and explain types of variables.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.1.2 from the Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;

- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on variables and types of variables.
- Ask students to do the application activity 4.1.2. and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.1.2

a) Clients' satisfaction and the amount of money saved.

b) Clients' satisfaction will give qualitative data, while the amount of money saved will give quantitative information.

e) Answers of application activity 4.1.2

i) Clients' satisfaction.

Since there is only a single variable of interest, this is univariate statistics.

ii) Clients' satisfaction and educational level.

Since there is only a single variable of interest, this is not a univariate statistics.

Lesson 3 : Data and types of data

a) Learning objective: Differentiate types of data.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this unit if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.1.3 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;

- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on data and types of data.
- Ask students to do the application activity 4.1.3. and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.1.3

Data are individual items of information that come from a population or sample. Data is also defined as a set of observations.

Example

You go to the supermarket and purchase three soft drinks (500ml soda, 1ml milk and 300ml juice) at 5000frw, four different kinds of fruits (apple, mango, banana and avocado) at 800frw, two different kinds of vegetables (broccoli and carrots) at 500frw, and two desserts (ice cream and biscuits) at 1000frw. The prices (5000frw, 800frw, 500frw, and 1000frw) are data in this example. Types of soft drinks, vegetable, fruits, and desserts are also data.

e) Answers of application activity 4.1.3

65000Frw, 45000Frw, 65000Frw, 15000Frw, 55000Frw, 35000Frw, 25000Frw, 45000Frw, 85000Frw and 95000Frw, are data which are quantitative.

Lesson 4 : Levels of measurement scale

a) Learning objectives: Differentiate levels of measurement scale

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.1.4 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;

- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills on levels of measurement scale.
- Ask students to do the application activity 4.1.4. and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.1.4

Variables involved in this research are types of places, satisfaction, and age.
Types of places and satisfaction are categorical, while age is numerical.
Types of places cannot be ranked or ordered, but satisfaction and age can be ranked.
Only for the age you can find the difference between its data values.

e) Answer of Application activity 4.1.4

i) Ages of the company workers (in years) is a ratio measurement scale.

ii) Color of clothes in a shop is a nominal measurement scale.

iii) Temperatures inside the room (in Celsius) is a ratio measurement scale.

iv) Nationalities of the company workers is a nominal measurement scale.

v) Salaries of the company employees is a ratio measurement scale.

vi) Weights of boxes of fruits is a ratio measurement scale.

Lesson 5 : Sampling and sampling methods

a) Learning objectives

Differentiate different sampling methods and decide which one is feasible depending on the context under study.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.1.5 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in sampling methods.
- Ask students to do the application activity 4.1.5. and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.1.5

The school may select a group of students (sample) and collect information on how much they spend.

When selecting students, the school should ensure the representativeness of students of all categories.

Answer for application activity 4.1.5

a) Systematic

b) Simple random sampling

c) Cluster

d) Stratified

e) Convenience

Lesson 6 : Frequency table

a) Learning objective

Represent data accurately using a frequency table.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.2.1 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in representing data using a frequency table.
- Ask students to do the application activity 4.2.1 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.2.1

The revenue paid by many people is 27000 FRW which four people paid.

Revenue paid (in FRW)	Frequency (number of people who paid each revenue)
21000	1
22000	1
23000	1
24000	2
25000	1
26000	1
27000	4
28000	1
29000	3
31000	2
34000	1
36000	1
39000	1

Answer for application activity 4.2.1

Marital Status	Frequency
Divorced	8
Married	11
Separated	10
Single	21
Total	50

Lesson 7 : Bar graph

a) Learning objective

Represent data accurately using a bar graph.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.2.2 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in representing data using a bar graph.
- Ask students to do the application activity 4.2.2 and evaluate whether lesson objectives were achieved to assess their competences.

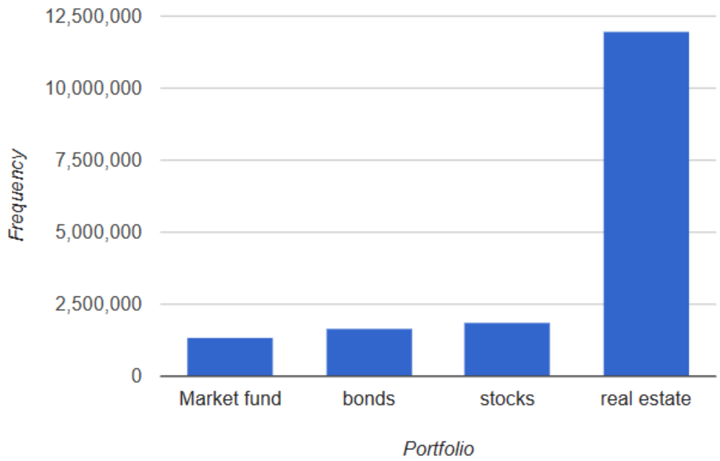
Answers of learning activity 4.2.2

a) Microsoft, other, Lotus Development, Novell, WordPerfect, Autodesk, Ashton-Tate, Borland International, Adobe Systems, Software Publishing, Aldus, Santa Cruz, Symantec

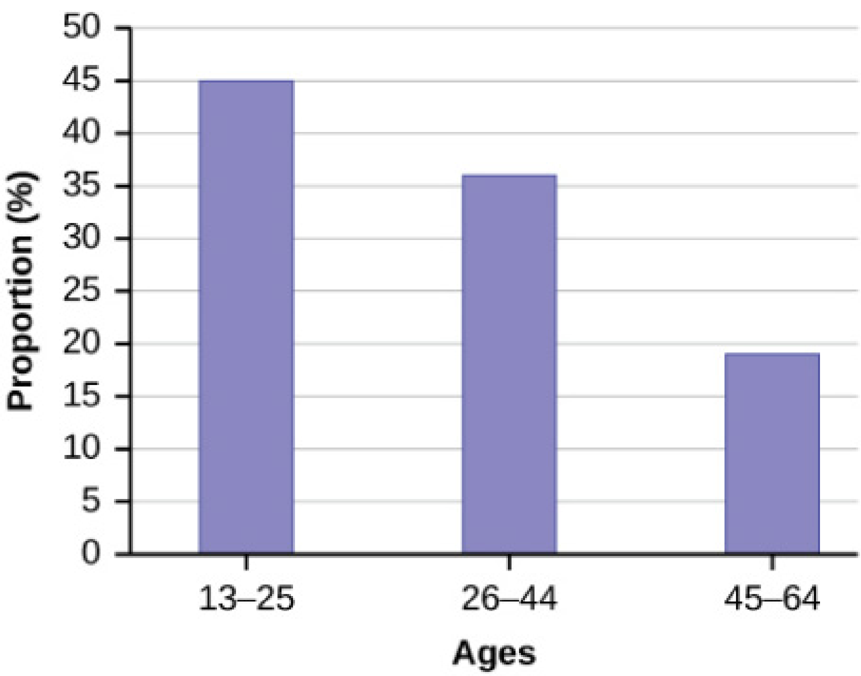
b) $\frac{19.5}{100} \times 5.7 = 1.11$ billion.

c) $\frac{46.3}{100} \times 5.7 = 2.64$ billion.

e) Answer of application activity 4.2.2



1)



2)

Lesson 8 : Histogram

a) Learning objective

Represent data accurately using a histogram.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.2.3 from Student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in representing data using a histogram.
- Ask students to perform the application activity 4.2.3 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.2.3

Nine people spend more hours at work. They spend between 20 and 25 hours per week.

a) 25 people

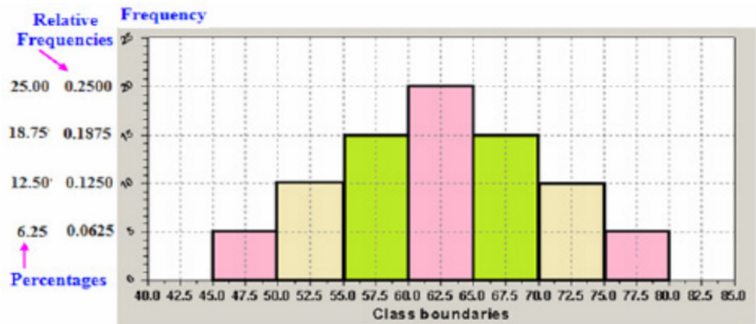
b) Histogram

e) Answer of application activity 4.2.3

a.

No of Class	Class Boundaries	Midpoint	Frequency	Relative Fre-quencies	Percentages	Ascending Cumulative Frequency
1	45-50	47.5	5	0.0625	6.25%	5
2	50-55	52.5	10	0.1250	12.50%	15
3	55–60	57.5	15	0.1875	18.75%	30
4	60–65	62.5	20	0.2500	25.00%	50
5	65–70	67.5	15	0.1875	18.75%	65
6	70–75	72.5	10	0.1250	12.50%	75
7	75–80	77.5	5	0.0625	6.25%	80
	Sum		**80**	**1**	**100%**	

b.



Lesson 9 : Time series graph

a) Learning objective

Represent data accurately using a time series graph.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.2.4 from the S4 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;

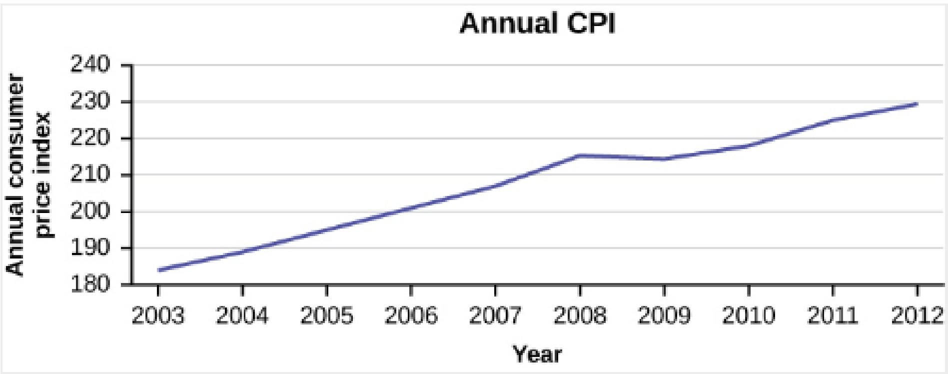
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in representing data using a time series graph.
- Ask students to do the application activity 4.2.4 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.2.4

a) In the fourth quarter of 2006

b) 24 people

c) 14+24+9+8+12+22+11+7=107 people



Lesson 10 : Pie chart

a) Learning objective

Represent data accurately using a pie chart.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.2.5 from the S4 Mathematics student's book;

- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in representing data using a pie chart.
- Ask students to do the application activity 4.2.5 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.2.5

a) The highest numbers of teenagers drink when upset (41%)

b) $(41\% + 25\%) = 66\%$ of teenagers drink because they are bored or upset.

c) Answer of application activity 4.2.5

$Number\ of\ mangoes = (100/360) \times 144 = 40$,

Number of $oranges = (70/360) \times 144 = 28$,

Number of $pawpaws = (40/360) \times 144 = 16$,

Number of $apples = (150/360) \times 144 = 60$.

$Cost\ of\ mangoes = 40 \times 30FRW = 1200FRW$,

$cost\ of\ pawpaws = 16 \times 160FRW = 2560FRW$.

The total cost of mangoes and pawpaws is $= 2560FRW + 1200FRW = 3760FRW$

a) The most unsold is Apple i.e. 60 apples

b) Frequency table

Type of fruits	Frequency (number remaining)
Mangoes	40
Oranges	28
Pawpaws	16
Apples	60
Total	144

Lesson 11 : Graph interpretation

a) Learning objective:

Interpret a statistical graph.

b) Teaching resources:

Manila papers, calculators, markers, student's book, pens, notebooks.

c) Prerequisites/Revision/Introduction:

Students will perform well in this unit if they make a good revision on the content of statistics learnt in senior three and in previous lessons of this unit.

d) Learning activities

- Invite students to work in groups and do the learning activity 4.2.6 from Student's books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a teacher, harmonize the findings from presentation highlighting elements to be verified on a graph of data;
- Use different probing questions and guide them to explore examples given in the student's book and lead them to discover the different ways of interpreting statistical data given graphically in reports or newspapers.
- After this step, guide students to do the application activity 4.2.6 and evaluate whether lesson objectives were achieved.

Answers of learning activity 4.2.6

a) There are 5 students with small size (S);

b) There are 13 students with medium size, 8 Students with large size and 4 students with Extra-large size.

e) Answers of application activity 4.2.6

a) There are 240 bags of cement produced in 8 minutes;

There are 96 bags of cement produced in 3min 12 seconds

There are 150 bags of cement produced in 5 minutes;

There 210 bags of cement produced in 7 minutes;

b) b. It will take 2 minutes 48 seconds to produce 78 bags of cement.

c)

Number of bags	96	150	210	240
Time in minutes	3min 12sec	5	7	8

Lesson 12 : Describing data using mean, median, and mode

a) Learning objective

Describe data using mean, median, and mode and interpret the results accurately.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.3.1 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in describing data using mean, median, and mode.
- Ask students to do the application activity 4.3.1 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.3.1

a) A portfolio's mean return on investment is an arithmetic average of returns achieved over specified periods. The statistic can easily be calculated by adding together all returns for a portfolio per unit of time and dividing by the number of observations. The mean return on investment would be calculated as follows:

$$\frac{(0.1-0.03+0.08+0.12+0.12-0.07+0.03)}{7}=0.05$$

This would give us a mean return of **5%** over the seven quarters.

b) We can arrange the return of the portfolio in the following ascending order: $Q_5=-7\%,\ Q_2=-3\%,\ Q_7=+3\%,\ Q_3=+8\%,\ Q_1=+10\%,\ Q_4=+12\%,\ and\ Q_6=+12\%.$

The middle value in this series is 8%, achieved in Q_3. Therefore, the median return of the portfolio would be 8%.

a) The return of the portfolio that has been achieved frequently is +12%, achieved in Q_4 and Q_6. Therefore, the mode would be +12%.

e) Answer of application activity 4.3.1

This is a sample of $n=6$, where

$$x_1=104,\ x_2=340,\ x_3=140,\ x_4=185,\ x_5=270,\ and\ x_6=258.$$

We find the sample mean by adding all the observations and dividing by 6:

$$(104+340+140+185+270+258)/6=216.166.$$

As in the problem salaries are in thousands then the mean is $216167FRW$

To find the median monthly salary, we need to arrange as follows: 104, 140, 185, 258, 270,340

As n is even (n=6), Median is given by

$$\frac{1}{2}\left[\left(\frac{6}{2}\right)^{th}+\left(\frac{6}{2}+1\right)^{th}\right]=\frac{1}{2}\left[3^{rd}+4^{th}\right]=\frac{1}{2}[185+258]=221.5$$

As in the problem salaries are in thousands then median is 221500FRW

Lesson 13 : Summarizing data using variance, standard deviation, range, mean deviation, and coefficient of variation

a) Learning objective

Summarize data using variance, standard deviation, range, mean deviation, and coefficient of variation and interpret the results accurately.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this unit if they are familiar with statistics learned in lower secondary school (S1 and S3).

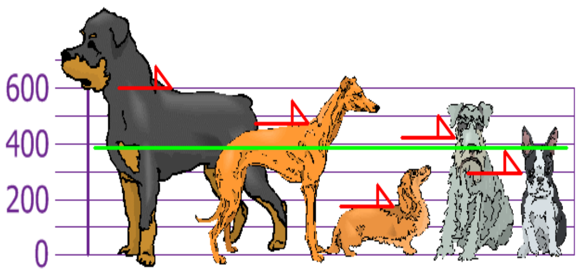
d) Learning activities:

- Invite students to work in groups and do learning activity 4.3.2 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in summarizing data using variance, standard deviation, range, mean deviation, and coefficient of variation.
- Ask students to do the application activity 4.3.2 and evaluate whether lesson objectives were achieved to assess their competences.

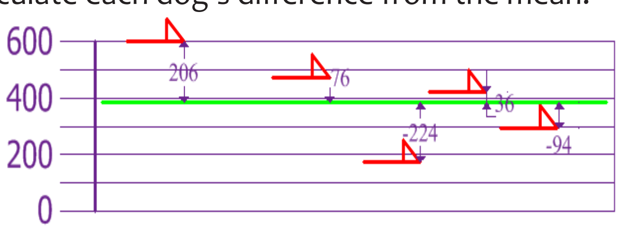
Answers of learning activity 4.3.2

The heights (at the shoulders) are 600mm, 470mm, 170mm, 430mm, and 300mm.

a) $\frac{\sum_{i=1}^{5} x_i}{5} = \frac{600 + 470 + 170 + 430 + 300}{5} = 394$. so, the mean (average) height is 394 mm. Let's plot this on the chart:



b) Now we calculate each dog's difference from the mean:



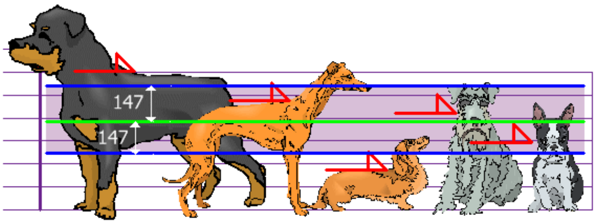
To calculate the Variance, take each difference, square it, and then average the result:

$$\sigma^2 = \frac{206^2 + 76^2 + (-224)^2 + 36^2 + (-94)^2}{5} = 21704$$

So the variance is **21704.**

Standard deviation is just the square root of the variance, so: $\sigma = \sqrt{21704} = 147.32$.

And the good thing about the standard deviation is that it is useful. Now we can show which heights are within one standard deviation (147mm) of the mean:



So, using the Standard Deviation, we have a "standard" way of knowing what is normal and what is extra-large or extra-small.

e) Answer of application activity 4.3.2

For pianos, the cost of the piano is 0.4 standard deviations BELOW the mean. For guitars, the guitar cost is 0.25 standard deviations ABOVE the mean. For drums, the cost of the drum set is 1.0 standard deviations BELOW the mean. Of the three, the drums cost the lowest compared to other instruments of the same type. The guitar costs the most compared to other instruments of the same type.

Lesson 14 : Determining the position of data value using quartiles

a) Learning objective

Determine the position of data value using quartiles and interpret the results accurately.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this unit if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.3.3 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in determining the position of the data value using quartiles.
- Ask students to do the application activity 4.3.3 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.3.3

i) The median, M, lies at the 6^{th} position and divides the set into two halves; thus, M = 5600.

ii) There are five items with prices below M and five above M.

iii) The middle value of the lower half lies in the third position, which is 4800.

iv) The middle value of the upper half lies in the 9^{th} position, which is 7700.

v) The two middle prices and the median divide the set into four quarters.

In short, the lower middle value is called the first (lower) Quartile, denoted by Q1. The middle value of the upper half is called the third (upper) Quartile, denoted by Q3. Thus: M = 56, Q1 = 48, Q3 = 77.

e) Answer of application activity 4.3.3

We begin by arranging the heights in ascending rank or order. The middle height lies in the 7th position. median = 163 cm.

To obtain median, we take $\frac{13+1}{2}=7$. This tells us that the median is in the 7th position which is 163 cm.

To obtain the lower quartile, we take $\frac{13+1}{4}=3.5$.

This means the lower quartile, Q1 is between the 3rd and the 4th position i.e. 3.5th position. Hence, $Q_1=\frac{1}{2}(160+161)=160.5$

To obtain Q3, we take $\frac{(13+1)\times 3}{4}=10.5$.

This means Q3 lies between the 10th and the 11th items i.e. 10.5th position.

Hence, $Q_1=\frac{1}{2}(164+165)=164.5$.

Lesson 15 : Skewness

a) Learning objective

Determine the skewness of data.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

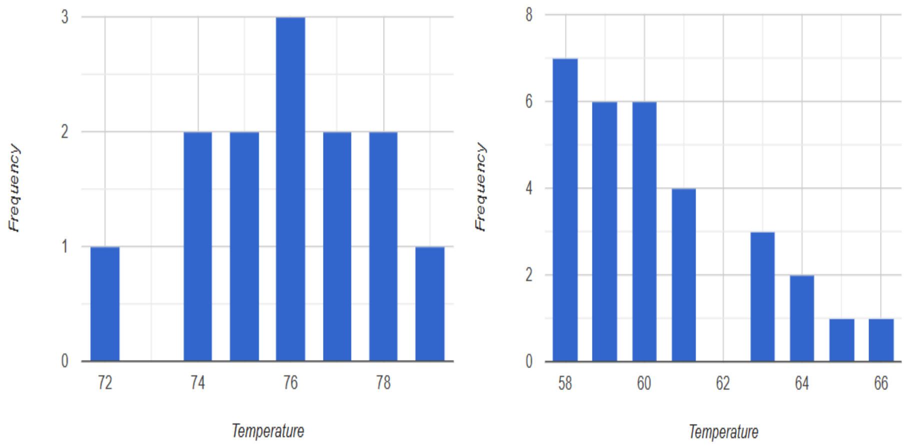
Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.4.1 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in determining the skewness of the data.
- Ask students to do the application activity 4.4.1 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.4.1

a.



b. Dataset 1: mean=75.9, median=76, and mode=76

Dataset 2: mean=60.4, median=60, and mode=58

c. Dataset 1: median=mode and both greater than mean

Dataset 2: mean> median> mode.

For dataset 1, all measures of a central tendency (mean, median, and mode) lie in the middle.

e) Answer of application activity 4.4.1

For histogram A, the data is skewed right, or there is a positive skewness.
The data is skewed left for histogram B, or there is a negative skewness.
For histogram A, the data is symmetric, or there is zero skewness.

Lesson 16 : Chebyshev's theorem and Empirical rule

a) Learning objective

Apply Chebyshev's theorem and Empirical rule to determine the skewness of the data.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.4.2 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in applying Chebyshev's theorem and Empirical rule to determine the skewness of the data.
- Ask students to do the application activity 4.5.2 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.4.2

Mean =73.4 and std=1.17

a. Eight prices fall within one standard deviation from the mean.

b. Two prices fall within two standard deviations from the mean.

c. None of the price that falls within three standard deviations from the mean.

Answer of application activity 4.4.2

a) According to the scale with measurements being given in grams: mean=74.06 and s=1.166=1.17

b) The correct readings are 79.2, 77.4, 78.3, 75.2, 77.5, 78.6, 77.2, 76.3, and 76. Therefore mean=77.26 and s=1.166=1.17.

If each reading is increased by 3.2, then the mean is increased by 3.2. The standard deviation, however, remains unaltered.

Lesson 17 : Examples of the applications of univariate statistics in solving problems related to finance, accounting, and economics

a) Learning objective

Apply univariate statistics in solving problems related to finance, accounting, and economics.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will perform well in this lesson if they are familiar with statistics learned in lower secondary school (S1 and S3).

d) Learning activities:

- Invite students to work in groups and do learning activity 4.5.1 from the S5 Mathematics student's book;
- Move around to facilitate students where necessary and give more clarification on eventual challenges they may face during their work;
- Ask randomly some groups to present their findings to the whole class;
- As a teacher, harmonize the group findings and use different probing questions to help students to explore examples and the content given in the student's book to enhance skills in applying univariate statistics in solving problems related to finance, accounting, and economics.
- Ask students to do the application activity 4.5.1 and evaluate whether lesson objectives were achieved to assess their competences.

Answers of learning activity 4.5.1

Answers will depend on what the student has provided.

Answer for application activity 4.5.1

1) a. 8 b. 535880FRW
2) 698000FRW

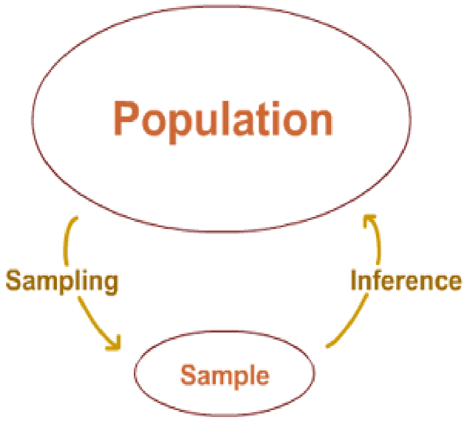
4.6 Summary of the unit

Statistics is the branch of mathematics that deals with the collection, organization, analysis, interpretation, and drawing of conclusions from numerical facts or data.

Statistics plays a vital role in nearly all businesses and forms the backbone for all future development strategies. Every business plan starts with extensive research, which is all compiled into statistics that can influence a final decision. Statistics helps the businessman to plan production according to the taste of customers.

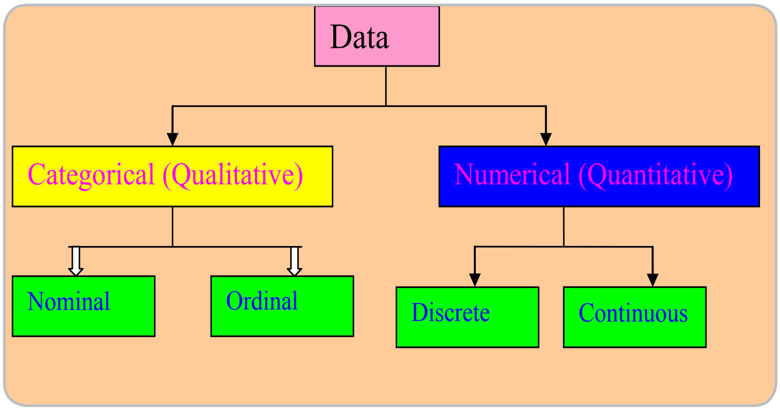
There are two branches of statistics, namely **descriptive statistics** and **inferential statistics**. **Descriptive** deals with describing the population under study. **Inferential** focuses on drawing conclusions about the population based on sample analysis and observation.

In statistics, we generally want to study a population. Because it takes a lot of time and money to examine an entire population, we select a sample to represent the whole population. The **population** is a collection of persons, things, or objects under study. A **sample** is the portion of the population that is available or to be made available for analysis.



For example, collecting the monthly savings data of every family that constitutes your population may be challenging if you are interested in the savings pattern of an entire country. In this case, you will take a small sample of families from across the country to represent the larger population of Rwanda. You will use this sample data to calculate its mean and standard deviation.

In statistics, we collect information on individual items from a population or sample, called statistical data. The collected data can be either qualitative or quantitative.



This unit also introduced some tabular and graphical representations of the data, such as frequency tables, histograms, bar graphs, time series graphs, and pie charts. We also looked at numerical descriptive measures: mean, median, mode, range, quartiles, variance, standard deviation, mean deviation, and coefficient of variation. Measures of symmetry were also discussed.

The following formulas are very important to describe data:

- Mean , $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, where n is the number of observations in the dataset, x_i are observations.

Or $\bar{x} = \frac{1}{n}\sum xf_i$ by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values.

- If n is odd, Median $= x_{\left(\frac{n+1}{2}\right)}$, or

median is given by $\left(\frac{n+1}{2}\right)^{th}$ number which is located on this position

- If n is even, Median $= \frac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}$, or

Median is given by $\frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$, then the median is a half of the sum of number located on those two positions.

- **The variance**

$\sigma^2 = \frac{\Sigma(x-\mu)^2}{N}$, where x is individual value, μ is the population mean, and N is the population size or

$S^2 = \frac{\Sigma\left(x-\bar{x}\right)^2}{n-1}$, where x is individual value, $\bar{x}$ is the sample mean, and n is the sample size.

- **Standard deviation**

$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\Sigma(x-\mu)^2}{N}}$, where x is individual value, μ is the population mean, and N is the population size, or

$S = \sqrt{S^2} = \sqrt{\frac{\Sigma\left(x-\bar{x}\right)^2}{n-1}}$, where x is individual value, $\bar{x}$ is the sample mean, and n is the sample size.

- **Coefficient of variation**

$CV = \frac{\sigma}{\mu}\times 100\%$, for population

$CV = \frac{S}{\bar{x}}\times 100\%$, for sample data

- **Quartiles**

In general, the lower quartile, Q_1 takes the $\left[\frac{1}{4}(n+1)\right]^{th}$ position from the lower end on the rank order. The upper quartile, Q_3 takes the $\left[\frac{3}{4}(n+1)\right]^{th}$ position on the rank order. For large population, it is enough to use $\left[\frac{1}{4}(\text{n})\right]^{th}$ and $\left[\frac{3}{4}(n)\right]^{th}$ positions for the lower and upper quartiles respectively.

- **Measure of skewness**

One measure of skewness, called Pearson's first coefficient of skewness, is to subtract the mean from the mode, and then divide this difference by the standard deviation of the data.

$$Skewness = \frac{Mean - Mode}{\sigma}$$

Pearson's second coefficient of skewness is also used to measure the asymmetry of a data set. For this quantity, we subtract the mode from the median, multiply this number by three and then divide by the standard deviation.

$$Skewness = \frac{3(Median - Mode)}{\sigma}$$

4.7 Additional Information for Teacher

While facilitating this unit, focus on interpreting the results obtained after computing statistical measures. Also, emphasize graphical interpretations.

When explaining Chebyshev's theorem and Empirical, lower the concept to secondary school students by focusing on symmetrical data. Don't talk too much about normal distribution because students will learn distribution in S6.

In this unit, while explaining, use many examples related to finance, accounting, economics as much as you can to help students understand concepts easily.

4.8. Answers of End unit assessment

This part provides the answers of end unit assessment activities designed in an integrative approach to assess the key unit competence with cross-reference to the student book.

1. Descriptive statistics.
2. Inferential statistics because you wish to generalize information collected from a sample to the whole population.
3. (a) Inferential statistics

   (b) Descriptive statistics
4. The variance is 5, and the standard deviation is 2.23.
5. The variance is 2.5, and the standard deviation is 1.58.
6. (a) 6 observations

   (b) 8 variables

   (c) Qualitative are 0 and quantitative are 8

4.9 Additional activities

4.9.1 Remedial activities

1. Eleven fish are sampled from a lake and their lengths were measured. The lengths, in centimeters, of the fish are: 17, 16, 10, 17, 17, 16, 14, 14, 16, 10, and 14. Calculate the variance and standard deviation.

   Answers: Sample variance is 6.65 and sample standard deviation is 2.58

2. Seven males are chosen randomly from a gym class in high school and are asked their shoe size. They are: 9, 12, 11, 9, 13, 10, and 10.What are the variance and standard deviation?

Answers: Sample variance is 2.28 and sample standard deviation is 1.51.

4.9.2 Consolidation activity

The mean of two samples of sizes 250 and 350 were 20 and 12, respectively. Their standard deviations were 2 and 5, respectively. Find the variance of combined sample of size 650.

Answer:

Combined mean= $\frac{n_1x_1+n_2x_2}{n_1+n_2}=\frac{250\times 20+320\times 12}{650}=13.6$

Let $d_1=x_1-combinedMean=20-13.6=6.4$, $d_2=x_2-combinedMean=12-13.6=-1.6$

$$\text{var}\,iance=\frac{\left[250\left(6.4+40.96\right)+320(13.6+2.56\right]}{650}=26.17$$

4.9.3 Extended activity

Suppose that a group of 100 students has an average age of 23 years, with a standard deviation of 2 years. What information does the empirical rule allow to obtain?

Answer: Construct the interval of the rule

Since the mean is 23 and the standard deviation is 2, then the intervals are:

- $\left[\mu-\sigma,\mu+\sigma\right]=\left[23-2,23+2\right]=\left[21,25\right]$
- $\left[\mu-2\sigma,\mu+2\sigma\right]=\left[23-4,23+4\right]=\left[19,27\right]$
- $\left[\mu-3\sigma,\mu+3\sigma\right]=\left[23-6,23+6\right]=\left[17,29\right]$

Calculate the number of students in each interval according to the percentages.

$(100)\times 68.27\% = 68$ students approximately.

$(100)\times 95.45\% = 95$ students approximately.

$(100)\times 99.73\% = 100$ students approximately.

Age intervals are associated with the numbers of students and interpret.

At least 68 students are between the ages of 21 and 25.

At least 95 students are between the ages of 19 and 27.

Almost 100 students are between 17 and 29 years old.

Unit 5 BIVARIATE STATISTICS AND APPLICATIONS

5.1 Key unit competence

Apply bivariate statistical concepts to collect, organise, analyse, present, and interpret data to draw appropriate decisions.

5.2 Prerequisite

Students will perform well in this unit if they are familiar with univariate statistics learned in the previous unit (Unit 4) and they are skilled in using scientific calculators.

5.3 Cross-cutting issues to be addressed.

- Inclusive education (promoting education for all in teaching)
- Peace and value Education (respect the views and thoughts of others during class discussions)
- Financial education (develop sprits of saving, and investments decisions)
- Gender (equal opportunity for boys and girls to participate in class)
- Environment sustainability (growth, etc)
- Standardized culture (quality of production)

5.4 Guidance on introductory activity

- In groups, facilitate students to read and do introductory activity from the student book.
- Guide students to read and analyse the questions insisting on the analysis of statistical data with two variables (x, y) and how they can interpret the bivariate data using correlation coefficients and regression lines.
- Facilitate any discussions, ensure that the work is done without noise and without unnecessary conversations.

- Walk around the classroom to assist students in need.
- Invite group representatives to present their findings and encourage gender in the presentation.
- In the lesson discussion, let students think of different ways to solve the problem.
- Basing on students' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

Answer of Introductory activity:

	X_i	Y_i	X_i^2	X_iY_i
	1	4	1	4
	2	8	4	16
	3	2	9	6
	4	12	16	48
	5	10	25	50
	6	14	36	84
	7	16	49	112
	8	6	64	48
	9	18	81	162
Σ	45	90		

$$X = \frac{1}{n}\sum X_i = \frac{45}{9} = 5\ ,\ X = \frac{1}{n}\sum y_i = \frac{90}{9} = 10$$

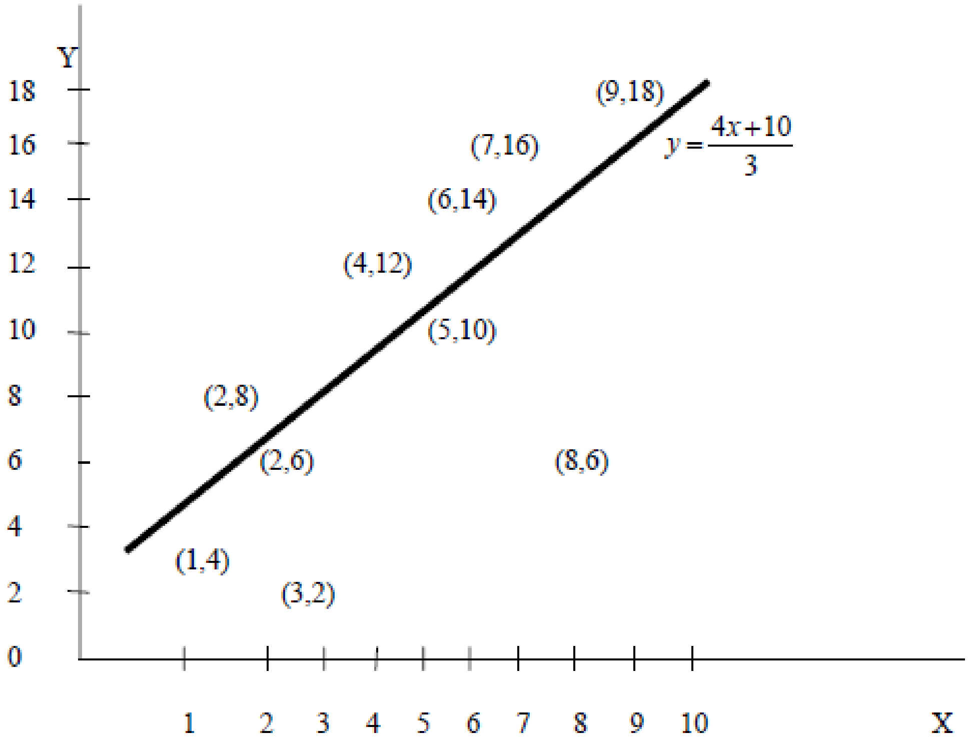
$$\boldsymbol{\delta}_{\mathbf{X,,Y}} = \frac{\Sigma X_i \Sigma Y_i - \frac{1}{n}\Sigma X_i \Sigma Y_i}{\Sigma X^2 - \frac{1}{n}(\Sigma X)^2} = \frac{530 - \frac{1}{9}(45)(90)}{285 - \frac{1}{9}(45)^2} = \frac{4}{3} = 1.33$$

The equation of regression line of

$$Y - Y_i = \delta_{X,,Y}(\mathrm{X} - \mathrm{X}_I)$$
$$Y - 10 = 1.33(\mathrm{X} - 5)$$
$$\mathrm{Y} = 10 + 1.33\mathrm{X} - 6.65$$
$$Y = 1.33X + 3.35$$

Scatter diagram: plotting the 9 sample points (1,4),(2,8),(3,4),(4,12),(5,10),(6,14),(7,16) (8,6),(9,18).

The first point on the line is (2,6) . Another point on the line is (x,y) = (5,10) so the regression line of y on x passes through the two points (2,6) and (4,10) plot these points and join them the required line of regression of is obtained.



5.5 List of lessons

Headings	#	**Lesson title/ sub-headings**	**Learning objectives**	**Number of periods (9)**
5.1 Introduction to bivariate statistics		Introductory activity	To arouse the curiosity of student-teacher on the content of unit 5.	1
	1	Key concepts of bivariate statistics	Differentiate between univariate data and bivariate data, independent variable and dependent variable, and represent graphically bivariate data in a scatter diagram.	1

5.2 Measures of the linear relationship between two variables and Applications: covariance, Correlation, regression line and analysis, and spearman's coefficient of correlation.	1	Covariance and correlation	Apply covariance and correlation coefficient in finding the relationship between two variables.	2
	2	Regression line and analysis	Apply regression line and analysis in estimating the value of one variable when one is known.	2
	3	Spearman's coefficient of correlation	Apply Spearman's coefficient of correlation to measure statistical dependence between two variables.	1
	4	Application of bivariate statistics	Apply bivariate statistics in accounting-related subjects.	1
5.3 End unit assessment				1

Lesson 1 : Key concepts of bivariate statistics

a) Learning objective:

Differentiate between univariate data and bivariate data, independent variable and dependent variable, and represent graphically bivariate data in a scatter diagram.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on univariate data and representation of points in x and y plane (cartesian plane).

d) Learning activities:

- Invite students to work in groups and do learning activity **5**.1.1 in their Mathematics books.
- Move around in the class for facilitating students where necessary and give more clarification.
- Verify and identify groups with different usable terminology.

- Invite one member from each group to present their work where they must explain what they have done in learning activity 5.1.1.
- As a teacher, harmonize the findings from presentation and guide students to answer the learning activity 5.1.1.
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define univariate data, bivariate data, independent variable and dependent variable. Use examples to differentiate univariate from bivariate statistics, independent variable from dependent variable.
- After this step, guide students to do the application activity 5.1.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 5.1.1

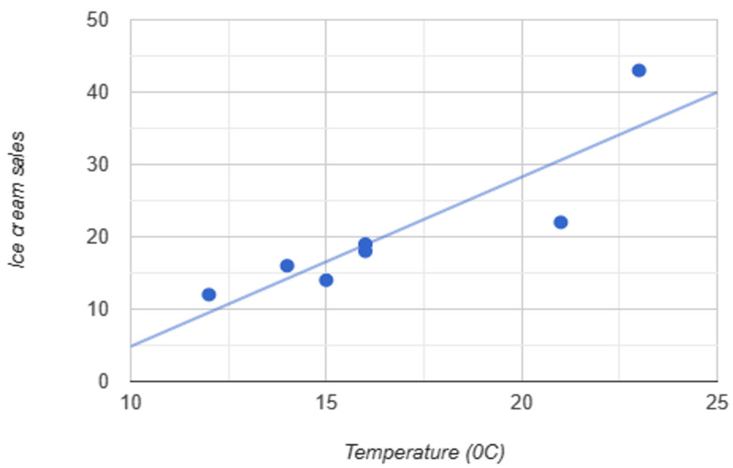
Use the learning activity to give an overview of the whole unit, the key concepts that will be discussed, and concepts such as Bivariate data, Scatter diagram, Covariance, Correlation, and Regression lines.

i) Statistical measure is a correlation. Variables are temperature and ice cream sales.

ii) Ice cream sales depend on temperature.

iii) Dependent variable (s)

iv)



v) According to the scatter diagram on iv, a positive relationship exists between temperature and ice cream sales. It shows an increase in temperature increased ice cream sales. This means that shops could use this information to buy more ice cream for hotter spells during the summer.

Answers for Application activity 5.1.1

1. Univariate statistics deals with one variable of interest whereas bivariate statistics deals with two variables of interest and how the two are related. If you are interested in finding out whether the more people save their income, the more financially stable they become, in this case you will need bivariate statistics. But if you are only interested in finding out the average income people save, in this case you will need univariate data.

2. Dependent variable is a variable that depends on the other. Independent variable is a variable that affect the other. From the example above on question 1, income will be an independent variable whereas becoming financially stable will be the dependent variable. In other words, to become financially stable depends on how much income you save.

Lesson 2 : Covariance and correlation

a) Learning objective:

Apply covariance and correlation coefficient in finding the relationship between two variables.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn this lesson better if they have a good background in univariate statistics.

d) Learning activities:

- Invite students to work in groups and do the activity 5.2.1 in their Mathematics books.
- Move around in the class for facilitating students where necessary and give more clarification to apply formula of covariance, and correlation in finding the relationship between two variables.

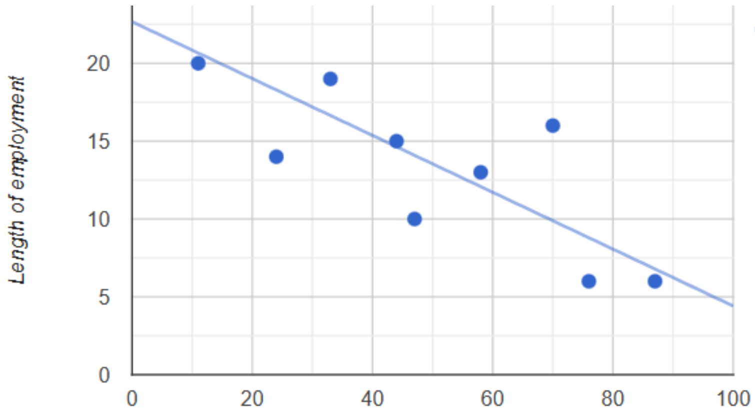
- Invite one member from each group to present and explain how to find relationship between two variables using covariance and correlation.
- As a teacher, harmonize the findings from presentation.
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to apply formula of covariance and correlation in finding relationship between two variables.
- After this step, guide students to the application activity 5.2.1 and evaluate whether lesson objectives were achieved.

Answers of learning activity 5.2.1

i) Correlation

ii) Covariance

Answers for application activity 5.2.1



There is a negative correlation.

Lesson 3 : Regression lines and analysis

a) Learning objective:

Apply regression line and analysis in estimating the value of one variable when one is known.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

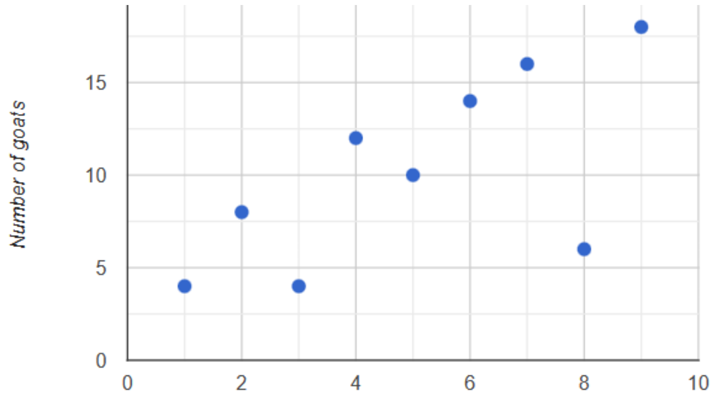
c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background in univariate statistics and linear equations.

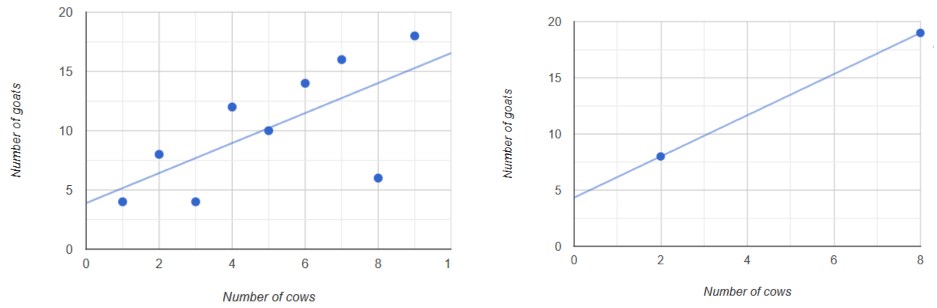
d) Learning activities:

- Invite students to work in groups and do the learning activity 5.2.2 in their Mathematics books.
- Move around in the class to facilitate students where necessary and give more clarification about how to apply the regression line and analysis in finding the unknown value of one variable when the other one is known.
- Monitor the work of different groups.
- Invite one member from each group to present their work, where they must explain the provided steps to find the regression lines.
- As a teacher, harmonize the findings from the presentation and guide students to find the regression lines and prediction of values of one variable when the values of the other one are known.
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to find the regression lines.
- After this step, guide students to application activity 5.2.2. and evaluate whether lesson objectives were achieved.

Answers for activity 5.2.2



a.



b.

c. Non-connected points are far from the straight line, they are scattered.

d. There is a positive relationship between the number of cows and a number of goats.

Answers for Application activity 5.2.2

a. $y = 0.19x - 8.098$

b. $y = 4.06$

Lesson 4 : Spearman's coefficient of correlation

a) Learning objectives

Apply Spearman's coefficient of correlation to measure statistical dependence between two variables.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background in univariate statistics covered in the previous unit.

d) Learning activities:

- Invite students to work in groups and do the learning activity **5.2.3** in their Mathematics books.
- Move around in the class to facilitate students where necessary and give more clarification to define and apply the formula of Spearman's coefficient of correlation.
- Invite one member from each group to present their work, where they must explain the provided steps to define and apply the formula of Spearman's coefficient of correlation in solving mathematical problems.

- As a teacher, harmonize the findings from the presentation and guide students to find Spearman's coefficient of correlation and apply that formula in solving mathematical problems.
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to apply Spearman's coefficient correlation in solving mathematical problems.
- After this step, guide students to the application activity 5.2.3 and evaluate whether lesson objectives were achieved.

Answers of learning activity 5.2.3

i) The death for underdeveloped countries (x) is 3 5 6 8 9 11.

The death rate for developed countries (y) is 2 3 4 5 6 8.

ii) See Rank (x) and Rank (y) in the table below.

iii)

Year	x	y	Rank (x)	Rank (y)	Rank (x)- Rank (y)=d	d2
1995	3	2	1	1	0	0
1996	5	3	2	2	0	0
1997	6	4	3	3	0	0
1998	8	6	4	5	-1	1
1999	9	5	5	4	1	1
2000	11	8	6	6	0	0
n=6						$\sum_{i=1}^{n} d_i^2 = 2$

$$1-\frac{6\times 2}{6\left(6^2-1\right)}=1-\frac{2}{35}=0.94\,.$$

Answer for Application activity 5.2.3

X	y	Rank (x)	Rank (y)	Rank (x)- Rank (y)=d	d²
12	6	6.5	3.5	3	9
8	5	2	2	0	0
16	7	9.5	5.5	4	16
12	7	6.5	5.5	1	1
7	4	1	1	0	0
10	6	4	3.5	0.5	0.25
12	8	6.5	7	0.5	0.25
16	13	9.5	10	0.5	0.25
12	10	6.5	8.5	2	4
9	10	3	8.5	5.5	30.25
					$\sum_{i=1}^{10} d_i^2 = 61$

$$1-\frac{6\times 61}{10\left(10^2-1\right)}=1-\frac{366}{990}=0.63.$$

Lesson 5 : Applications of bivariate statistics in accounting-related subjects or any other area: Relationship between advertising products and total revenue, the relationship between attendance in class and the marks scored, etc.

a) Learning objectives

Apply bivariate statistics in accounting-related subjects or any other area.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background in univariate statistics covered in the previous unit.

d) Learning activities:

- Invite students to work in groups and do the learning activity **5.2.4** in their Mathematics books.
- Move around in the class to facilitate students where necessary and give more clarification on applications of bivariate statistics in economics, accounting, biology, medicine, etc.

- Invite one member from each group to present their work, where they must explain the application of bivariate statistics.
- As a teacher, harmonize the findings from the presentation and explain how bivariate statistics is applied to different areas. You can refer to some examples provided in the student book.
- After this step, guide students to the application activity 5.2.4 and evaluate whether lesson objectives were achieved.

Answers of learning activity 5.2.4

Bivariate statistics can help predict the value for one variable if we know the value of the other.

Bivariate statistics can help find the relationship between two variables.

Answer for application activity 5.2.4

a) i. -0.976 ii. -0.292

b) The transport manager's order is more profitable for the seller, saleswomen is unlikely to try to dissuade.

c) (i) No, maximum value is 1. (ii) Yes, higher performing cars generally do less mileage to the gallon. (iii) No, the higher the engine capacity, the dearer the car.

d) When only two rankings are known; when relationship is non-linear.

5.6 Summary of the unit

Bivariate statistics deals with the collection, organization, analysis, interpretation, and drawing of conclusions from bivariate data.

Bivariate statistics plays a vital role in predicting one variable's value when the other one is known. We normally collect bivariate data to try and investigate the relationship between the two variables and then use this relationship to inform future decisions. In the case of bivariate data sets, we are often interested in whether elements with high values of one of the variables also have high values of other variables.

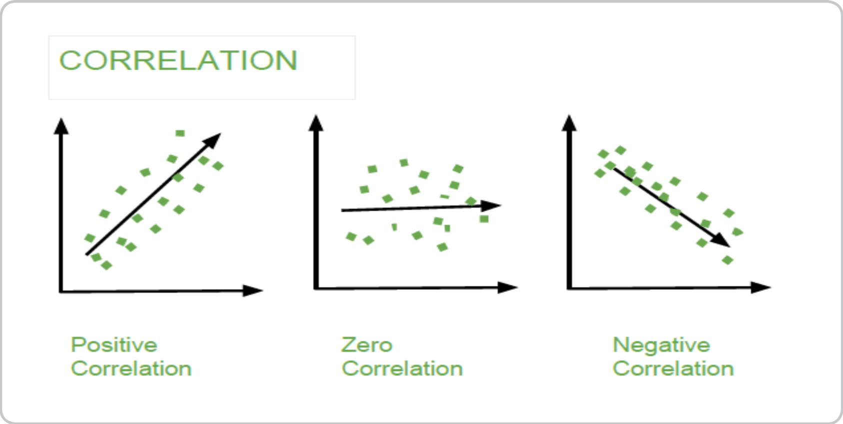
Data sets that contain two variables, such as wage and gender, and consumer price index and inflation rate data are said to be bivariate. For example, we may collect the monthly savings and number of family members data of every family that constitutes the population if we are interested in finding the relationship between savings and the number of family members. In this case, we will take a small sample of families from across the country to represent the larger population of Rwanda. We will use this sample to collect

data on family monthly savings and the number of family members.
This unit also introduced the graphical representation of bivariate data, called a scatter diagram. We also looked at numerical measures of the relationship between two variables: covariance and correlation. Regression lines were also discussed.
In this unit, each concept was introduced using a learning activity that aroused students' interest and curiosity about what was going to be learnt in the lesson, and the lesson ended with an application activity to check whether learning objectives were achieved or not.

5.7 Additional Information for Teacher

While facilitating this unit, focus on the difference between covariance and correlation and their interpretations.
The coefficient of correlation between two variables x and y is given by

$r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$, where $\operatorname{cov}(x, y)$ is covariance of x and y, σ_x is the standard deviation for x, σ_y is the standard deviation for y. Correlation describes how the two variables are related, whereas correlation describes how the two variables covary. The correlation coefficient ranges from -1 to +1.
If the linear coefficient of correlation takes values closer to -1, the correlation is strong and negative and will become stronger the closer r approaches -1.
Focus also on interpreting correlation based on how data are scattered in a scatter diagram.



5.8. Answers of End unit assessment

This part provides the answers of end unit assessment activities designed in an integrative approach to assess the key unit competence with cross-reference to the student book.

1. 0.26
2. 0.43. Some agreement between average attendance ranking a position in the league and a high position in the league correlating with high attendance.
3. $y = 0.61x + 10.5$, $x = 1.47y - 1.14$, $y = 28.83$
4. $y = 0.94x + 92.26$, blood pressure=134.56

5.9 Additional activities

5.9.1 Remedial activities

Identify positive and negative correlation situation in the following cases:

a) The more times people have unprotected sex with different partners, the more the rates of HIV in a society.

b) The more people save their incomes, the more financially stable they become.

c) As weather gets colder, air conditioning costs decrease.

d) The more alcohol is consumed, the less the judgment one has.

e) The more one cleans the house, the less likely are to be pests problems.

Answer:

a) Positive correlation

b) Positive correlation

c) Negative correlation

d) Negative correlation

e) Negative correlation

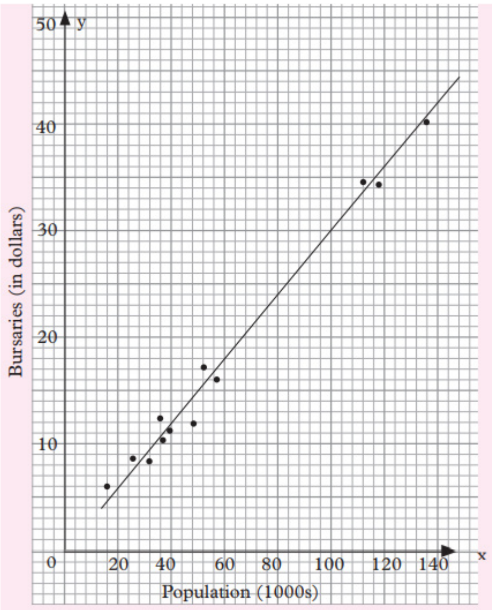
5.9.2 Consolidation activities

The amount of government bursaries allocated to certain administration regions in the country in a certain year is listed together with their population sizes.

Region	Population (10,000s)(x)	Bursaries in millions (y)
1	29	8.0
2	58	16.8
3	108	33.9
4	34	10
5	115	34
6	19	6.5
7	136	40.5
8	33	10.2
9	25	8.8
10	47	12.5
11	49	17.3
12	33	12.6

Draw a scatter diagram and assess whether there appears to be a correlation between the two measurements labeled x and y. Use a scale of 1 cm: 50 000 on the vertical axis and 1 cm: 10 m on the horizontal axis.

Answer:



The best line of fit in this scatter diagram follows the general trend of the points. This line has a positive gradient. Thus, the relation in this data is called a positive correlation.

5.9.3 Extended activities

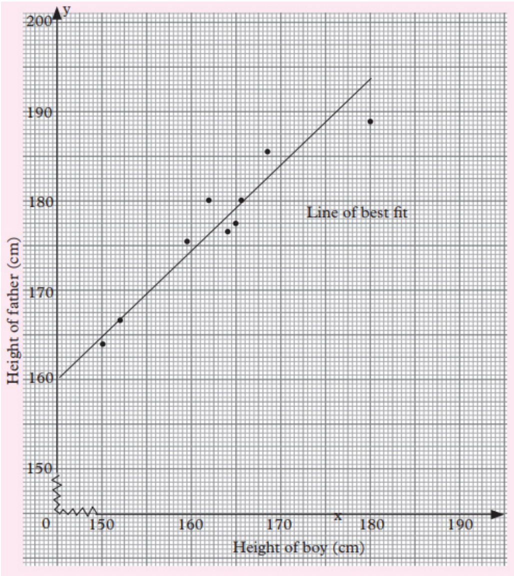
The following measurements were made, and the data was recorded to the nearest cm.

Height of boys (cm)	Height of his father (cm)
164	171
168	186
150	164
162	180
159	176
165	177
187	192
152	167
180	189
166	180

a) Plot the data using coordinate axes.
b) Draw the line of best fit for the data
c) Find the equation of the line in (b) above.
d) Describe the correlation.
e) Would it be reasonable to use your graph to estimate the height of a father whose son is 158 cm tall?

Answer:

(a) and (b)



c) Using points (152, 167) and (180, 189):

Gradient of line $\frac{189-167}{180-152}=\frac{11}{14}$

$y=mx+c$

From the graph c=160.

$$y=\frac{11}{14}y+160$$

Equation of the line of best fit can be written as $14y=11x+2240$.

d) The correlation in this data is positive or direct. The line of best fit has a positive gradient and, therefore, a positive trend.

e) Yes Height of father is 175 cm.

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